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Expected Cost Analysis of Probabilistic Programs

Research Seminar WS21/22

Jonas Schöpf

November 24, 2021

Motivation

- model natural/physical processes $\Rightarrow$ “real” coin flip

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- cryptography $\Rightarrow$ primality tests
- robotics
- machine learning
- improvement of algorithms, e.g., quicksort

Motivation - Quicksort

- “standard” vs. randomized quicksort

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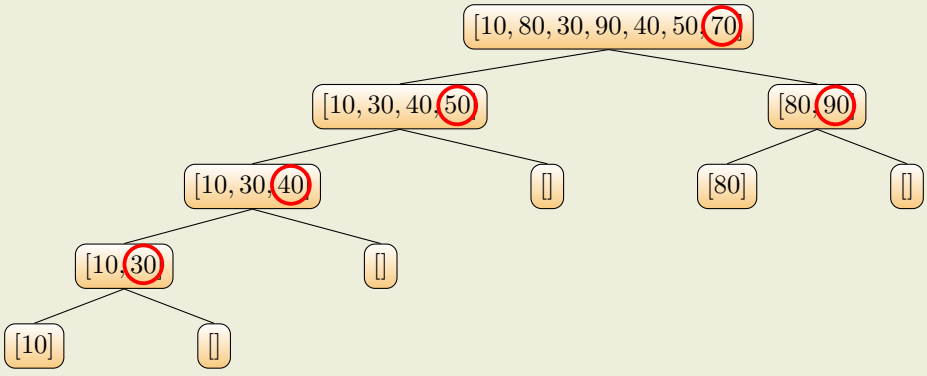
Example Quicksort



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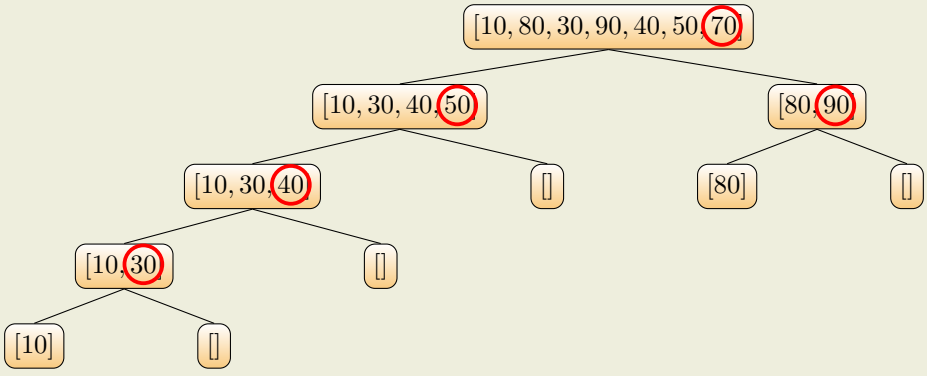

Motivation - Quicksort

- "standard" vs. randomized quicksort
- *first* vs. *last* vs. *random* vs. *median* pivot element
- worst case: $\mathcal{O}(n^2)$ vs. $\mathcal{O}(n^2)$ (BUT expected or average time complexity is $\mathcal{O}(n \log n)$)

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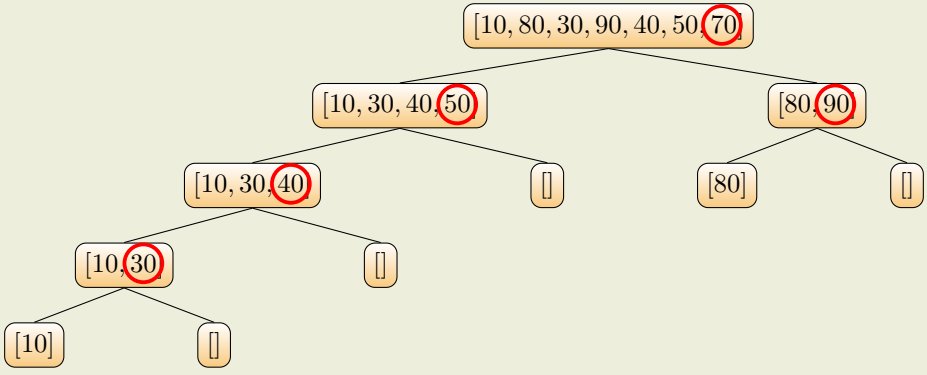

Overview

- Primer
- ERT
- ECT – Syntax & Semantic
- Automation
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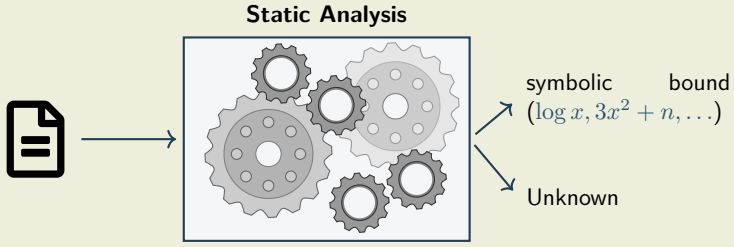
Static Resource Analysis



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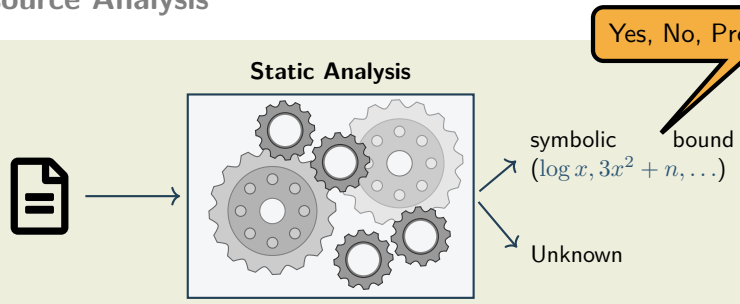

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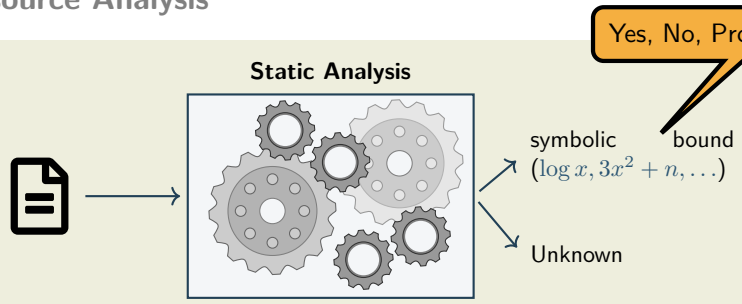

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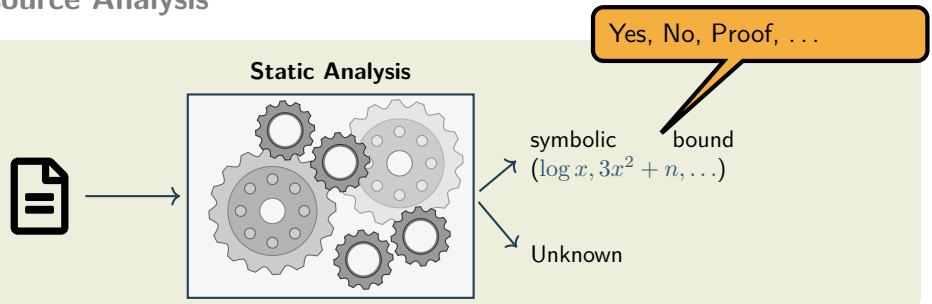

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- recurrence relations
- type systems
- term rewriting
- . . .

Non-/Deterministic vs. Probabilistic

	Non-/Determi.	Probabilistic
Dynamics		
Semantics		

Non-/Deterministic vs. Probabilistic

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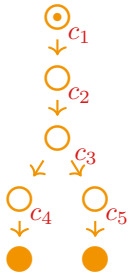

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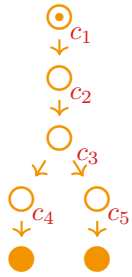

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- assign cost c_i to each operation
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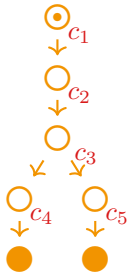

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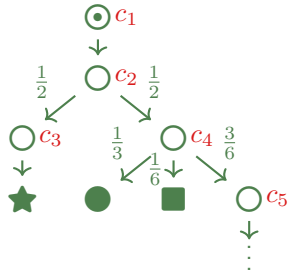

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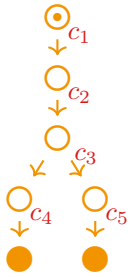

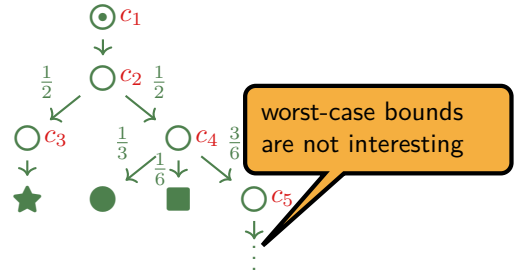

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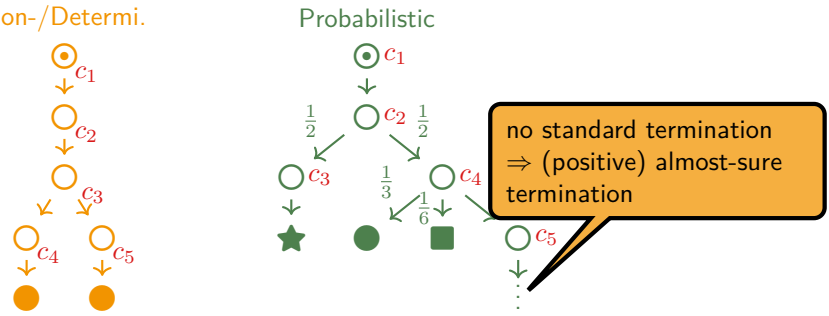

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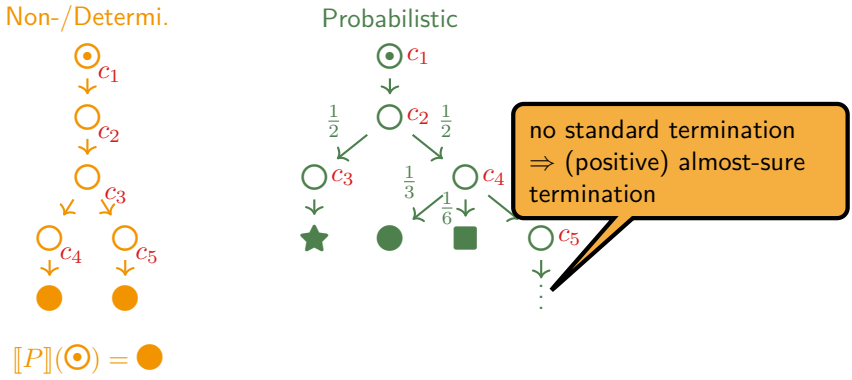

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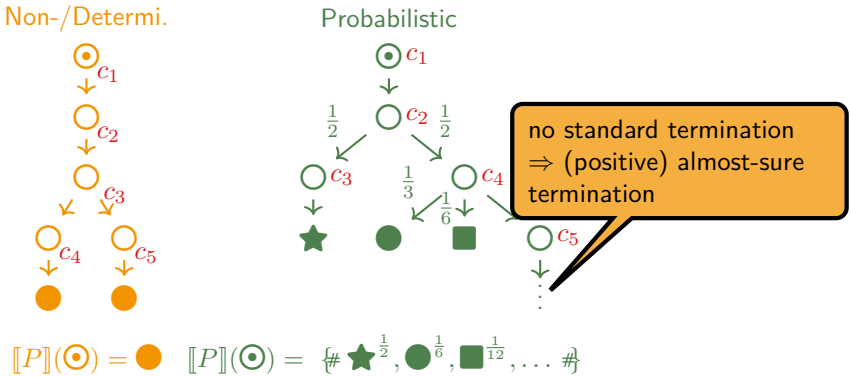

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- obtains bounds for the expected runtime of randomized algorithms and proves positive almost-sure termination
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Example

$$\mathbb{C}^{\Sigma} \triangleq \{t \mid t : \Sigma \to \mathbb{R}_{\geq 0}\}$$

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What Do We Want to Achieve?

We would like to have a calculus which to determine the expected runtime of a probabilistic program or algorithm.

- compositional
- modular
- precise

Furthermore it would be beneficial if termination follows from this calculus.

Probabilistic While (`pWhile`)

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Syntax of pWhile

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\begin{aligned}
\mathtt{C}, \mathtt{D} ::= \\
&\mid \mathtt{skip} \\
&\mid \mathtt{abort} \\
\\
&\mid \mathtt{C}; \mathtt{D} \\
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$\texttt{rand}(e), \texttt{unif}(n,m), \texttt{ber}(n,m), \ldots$

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Example – geo

$$
\begin{aligned}
& b := 1; \\
& x := 1; \\
& \texttt{while}(b = 1)\ \{ \\
& \quad \texttt{consume}(1); \\
& \quad x := x * 2; \\
& \quad b := \texttt{ber}(1,1)\}
\end{aligned}
$$

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We define the expected cost transformer (ECT) operating on cost functions over states. Thus $\mathsf{ect}[\mathtt{C}](f)$ can be seen as the cost of $\mathtt{C}$ w.r.t. a continuation cost f.

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$\{\mathtt{C}\} <> \{\mathtt{D}\}$	$\max(\mathsf{ect}[\mathtt{C}](f), \mathsf{ect}[\mathtt{D}](f))$	$\max(\mathsf{evalue}[\mathtt{C}](f), \mathsf{evalue}[\mathtt{D}](f))$

Probabilistic Abstract Reduction Systems (PARS)

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1. $\mathtt{ect}[\to](f)(\bullet \rhd \mathtt{C}) = \mathtt{ect}[\mathtt{C}](f)$
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Define $I = [b = 1] \cdot 2$.

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Define $I = [b = 1] \cdot 2$. Thus we have to check $b = 1 \models \mathtt{ect}[\mathtt{C}_{\mathsf{geo}}](I) \leq I$ and $b \neq 1 \models 0 \leq I$.

Consider $\mathtt{C}_{\mathtt{geo}}$ denoting the body of geo and $I = [b = 1] \cdot 2$ the upper invariant:

Example – geo

$$
\begin{array}{l}
b := 1; \\
x := 1; \\
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For two nested while loops $\texttt{while}(\phi)\ \{\texttt{while}(\psi)\ \{\texttt{C}\}\}$, we cannot synthesize the inner (I) and outer (O) upper invariant independently.

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Modularity

We achieve modularity by first calculating the cost of one loop iteration and then analyze how possible values in a state evolve during execution.

Separating Expected Cost and Value

First we have to separate the analysis into a cost and value analysis:

$$\mathsf{ect}[\mathtt{C}](f) \leq \mathtt{ecost}[\mathtt{C}] + \mathsf{evalue}[\mathtt{C}](f)$$

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We need all this for the automation of ECT.

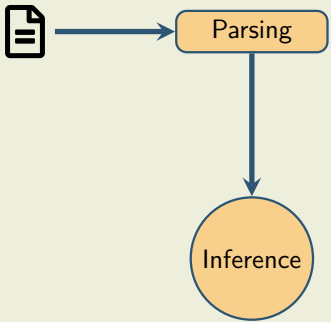
Overview

- Primer
- ERT
- ECT – Syntax & Semantic
- Automation
- Summary

ecoimp


Parsing

ecoimp



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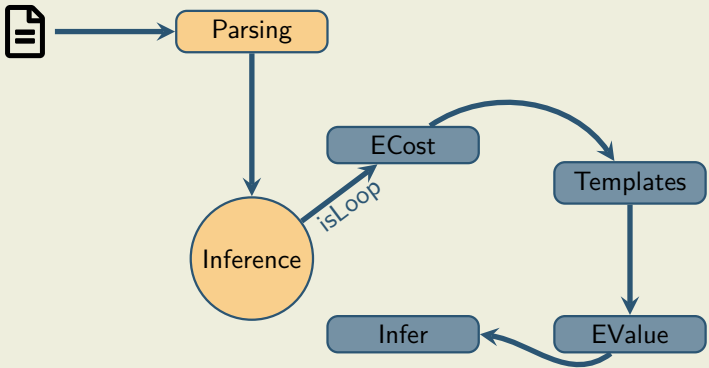

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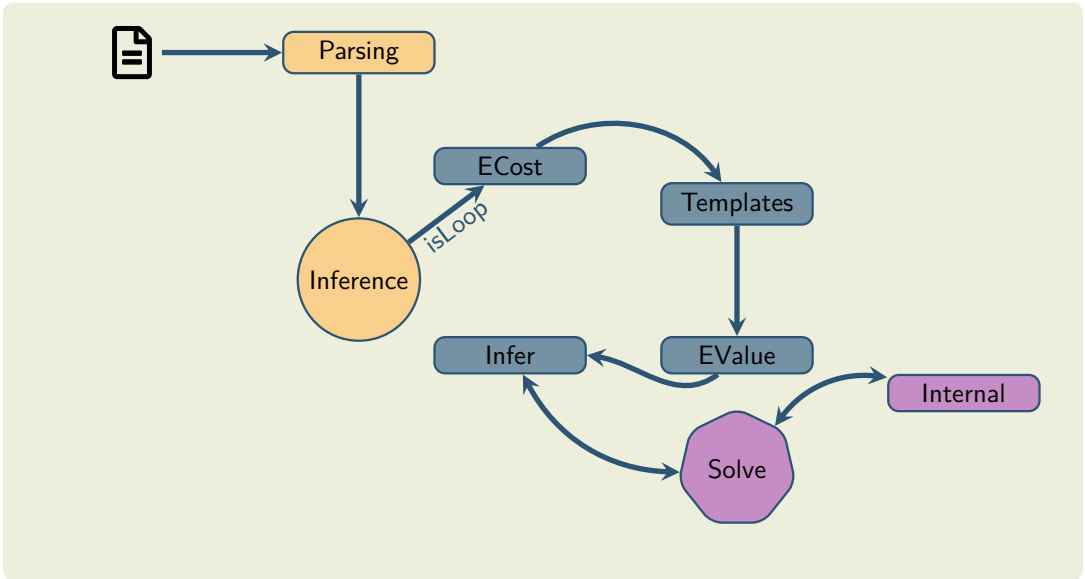
Parsing
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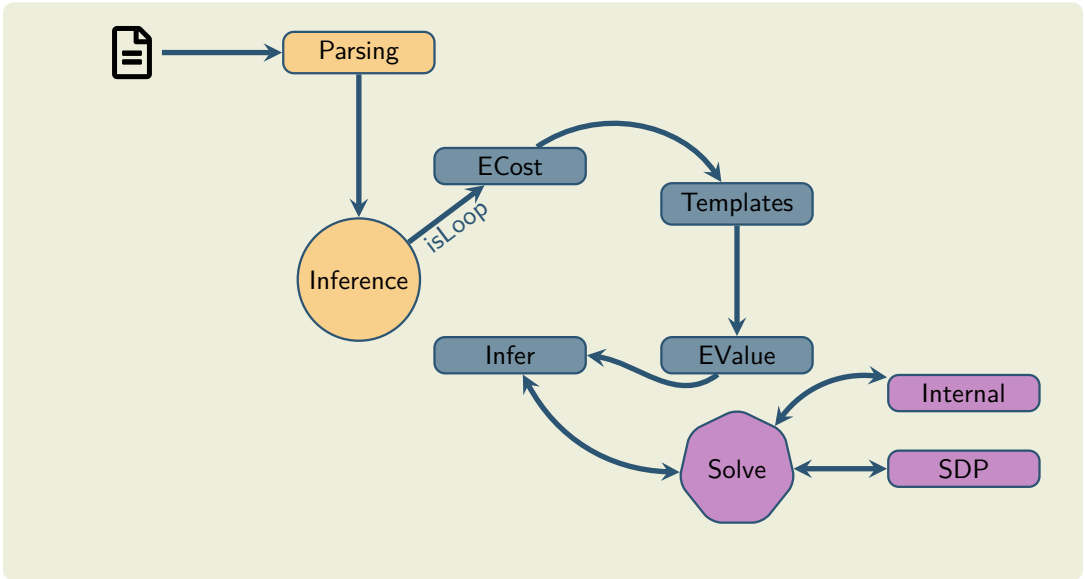
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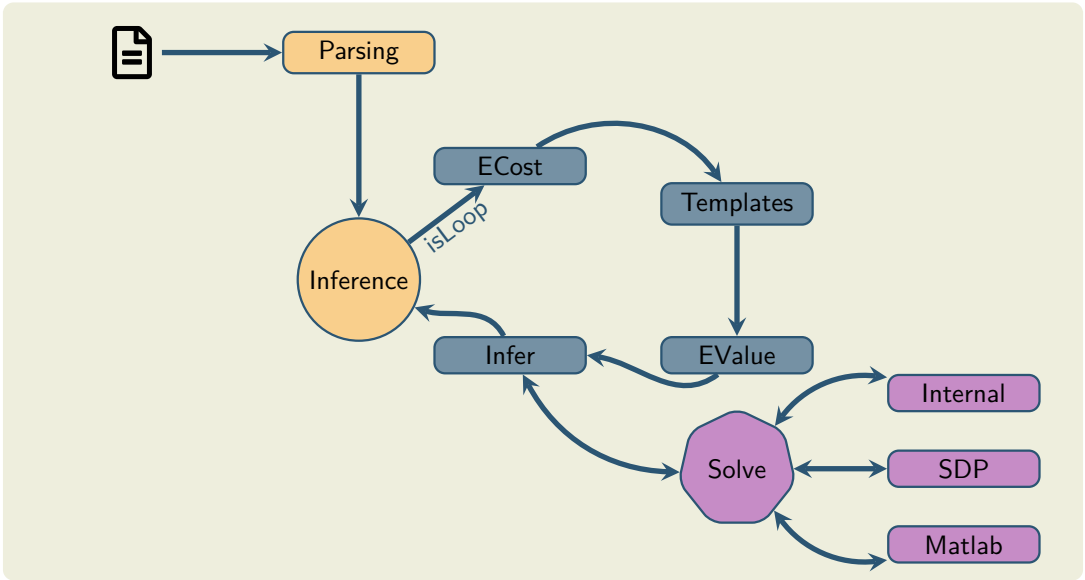
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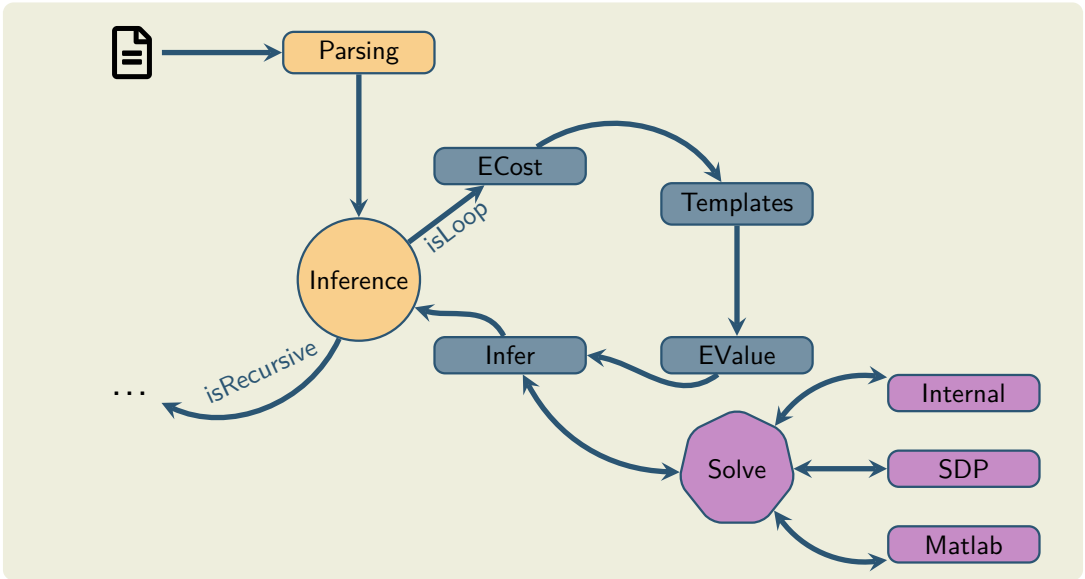
Parsing
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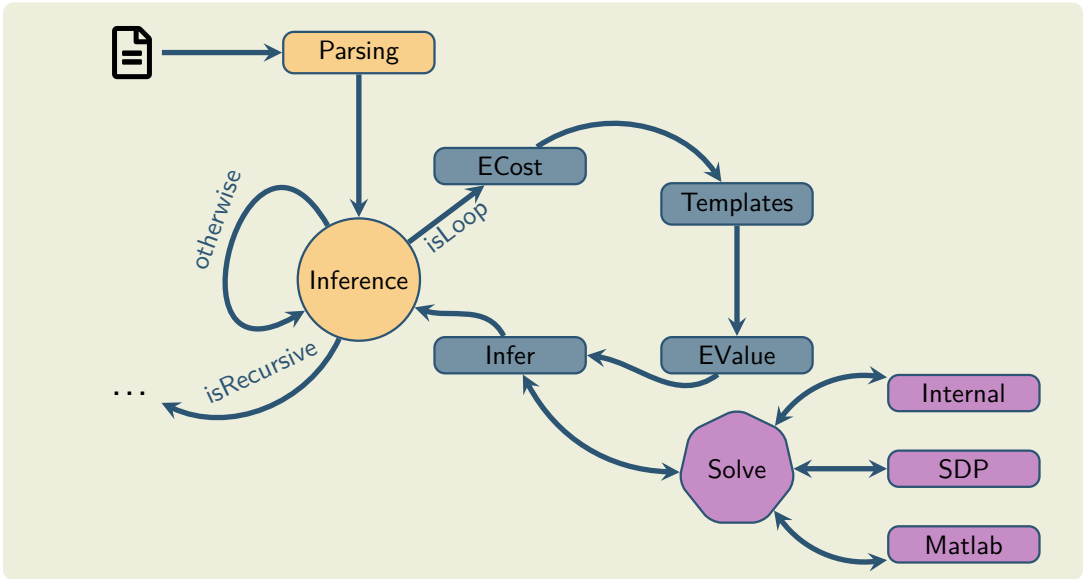
Templates
isLoop
Inference
Infer
EValue
isRecursive
...
Internal
Solve
SDP
Matlab

ecoimp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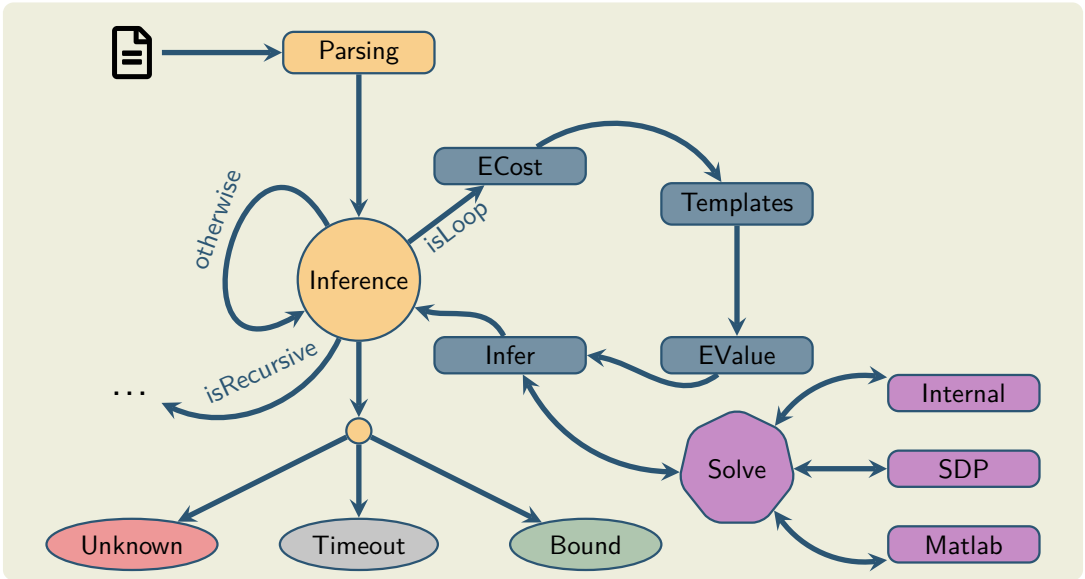
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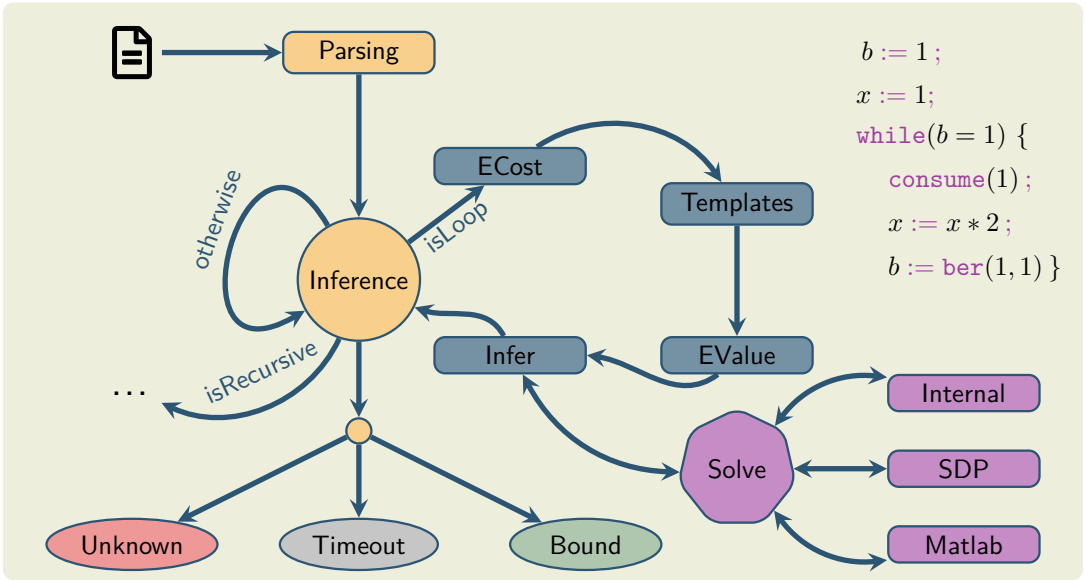
Unknown
Timeout
Bound

ecoimp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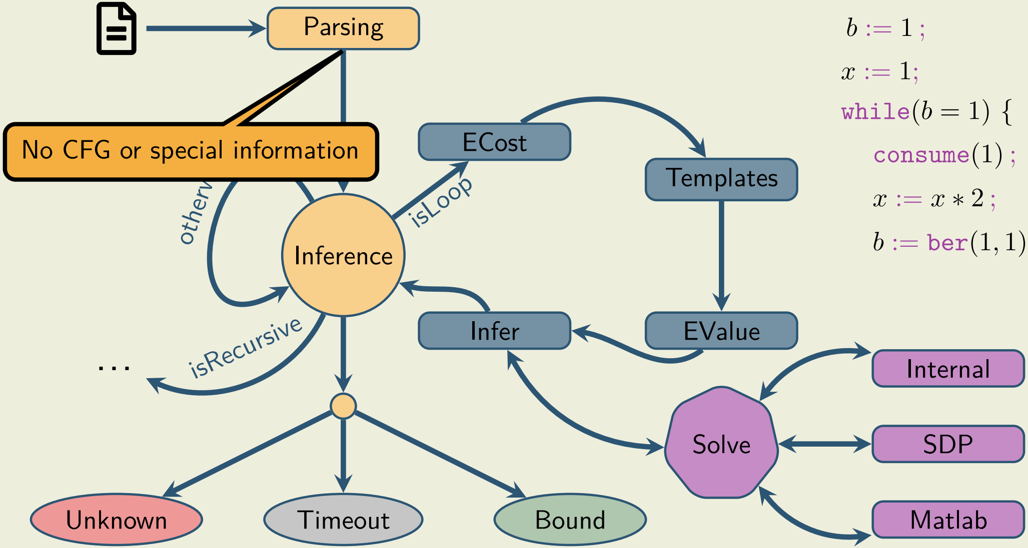
Timeout
Bound
$b := 1\,;$
$x := 1;$
$\texttt{while}(b = 1)\ \{$
$\texttt{consume}(1)\,;$
$x := x * 2\,;$
$b := \texttt{ber}(1,1)\ \}$

ecoimp



$b := 1\,;$

$x := 1;$

$\texttt{while}(b = 1)\ \{$

$\quad \texttt{consume}(1)\,;$

$\quad x := x * 2\,;$

$\quad b := \texttt{ber}(1,1)\ \}$

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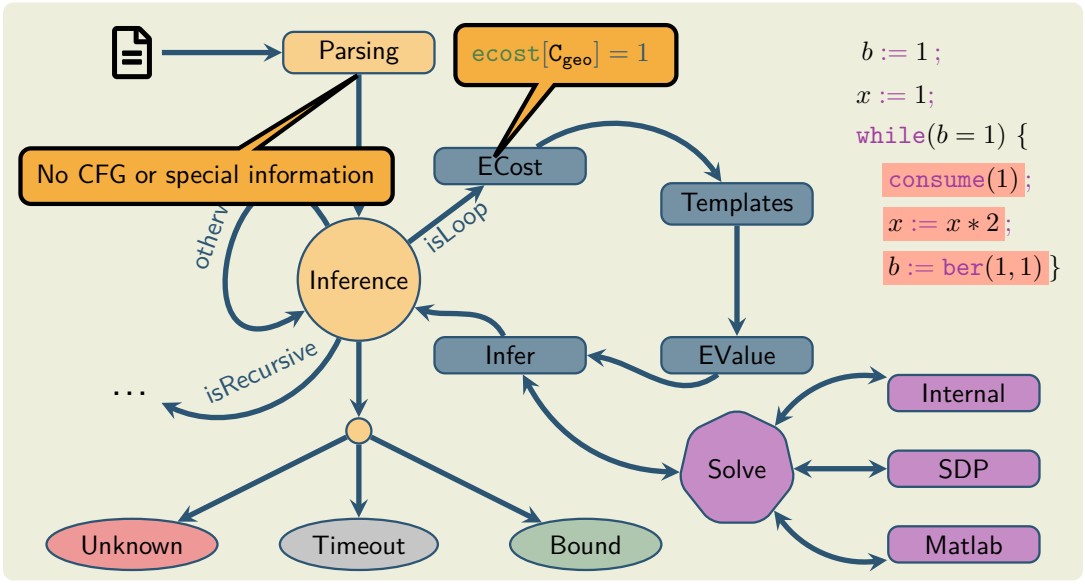
Parsing
ecost[C_geo] = 1
No CFG or special information
ECost
Templates
otherw
isLoop
Inference
isRecursive
...
Infer
EValue
Internal
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SDP
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Timeout
Bound
b := 1;
x := 1;
while(b = 1) {
consume(1);
x := x * 2;
b := ber(1,1)}

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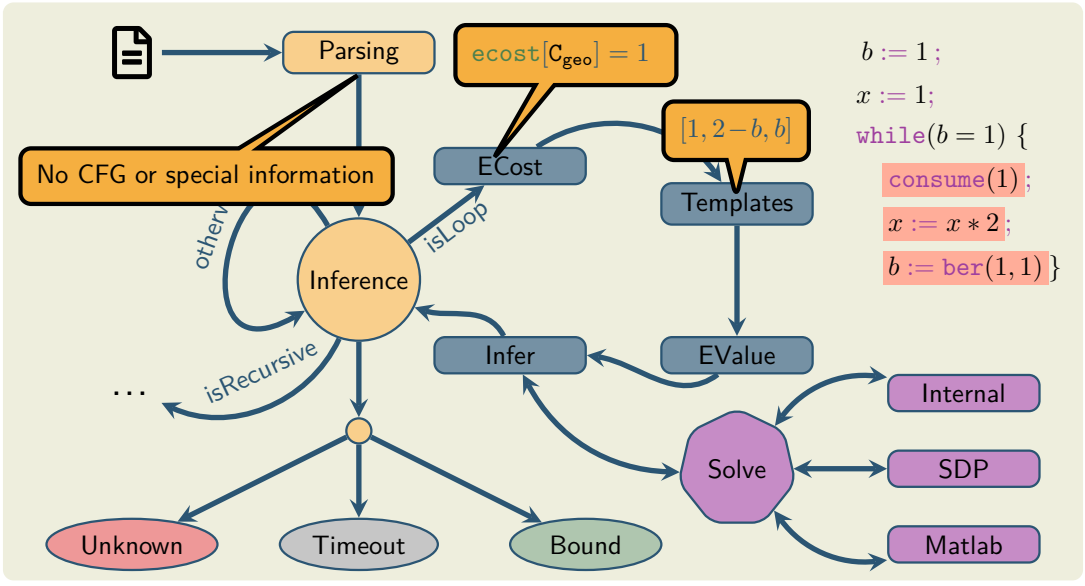
Parsing
ecost[C_geo] = 1
No CFG or special information
ECost
[1, 2−b, b]
Templates
otherw
isLoop
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Infer
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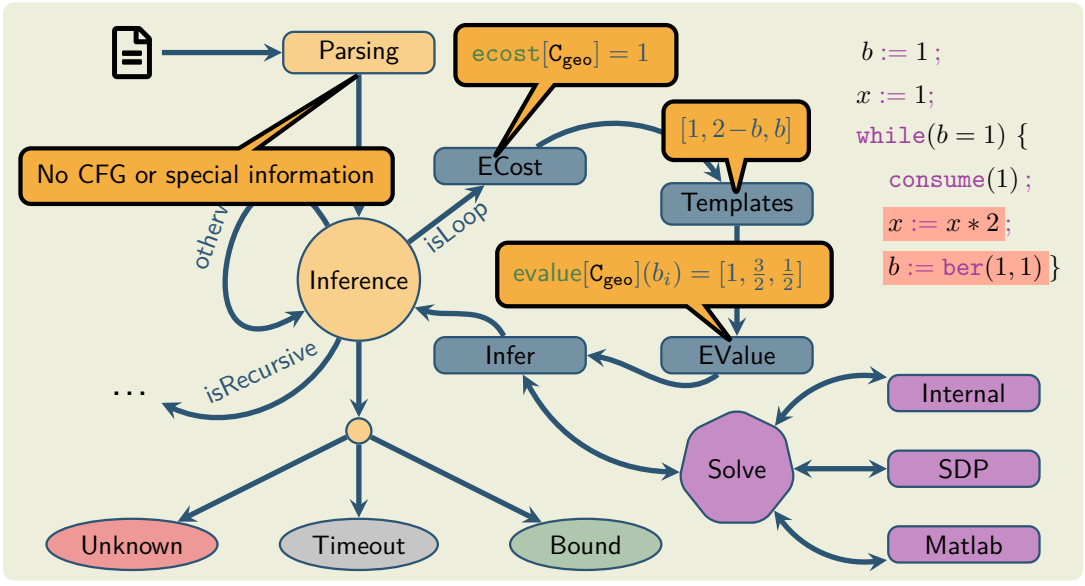
b := 1;
x := 1;
while(b = 1) {
consume(1);
x := x ∗ 2;
b := ber(1, 1)}

ecoimp


Parsing
ecost[C_geo] = 1
No CFG or special information
ECost
[1, 2−b, b]
Templates
otherw
isLoop
Inference
evalue[C_geo](b_i) = [1, 3/2, 1/2]
EValue
Infer
isRecursive
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Bound
b := 1 ;
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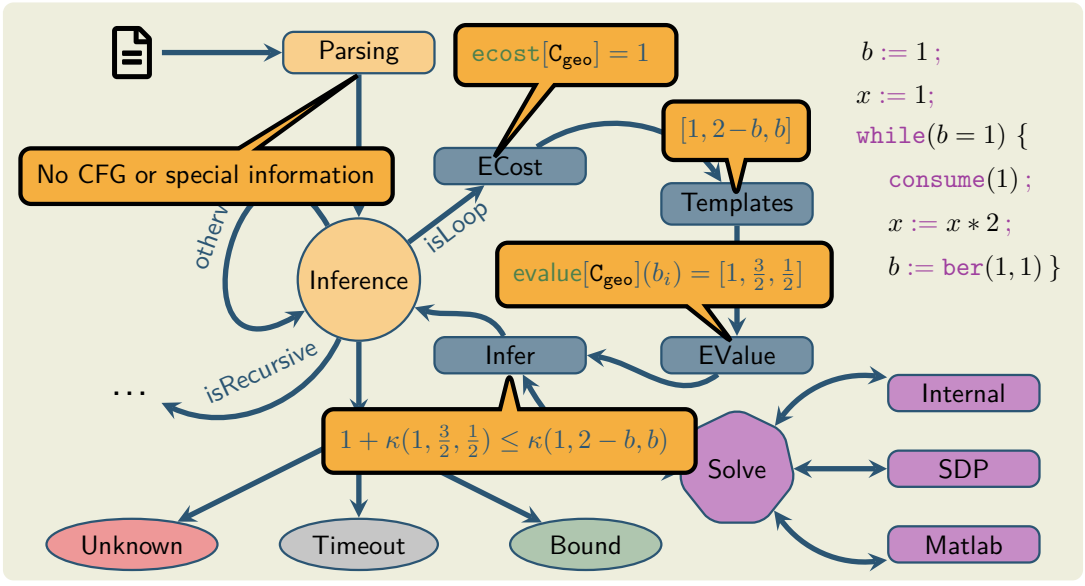
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1 + κ(1, 3/2, 1/2) ≤ κ(1, 2 − b, b)
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SDP
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Timeout
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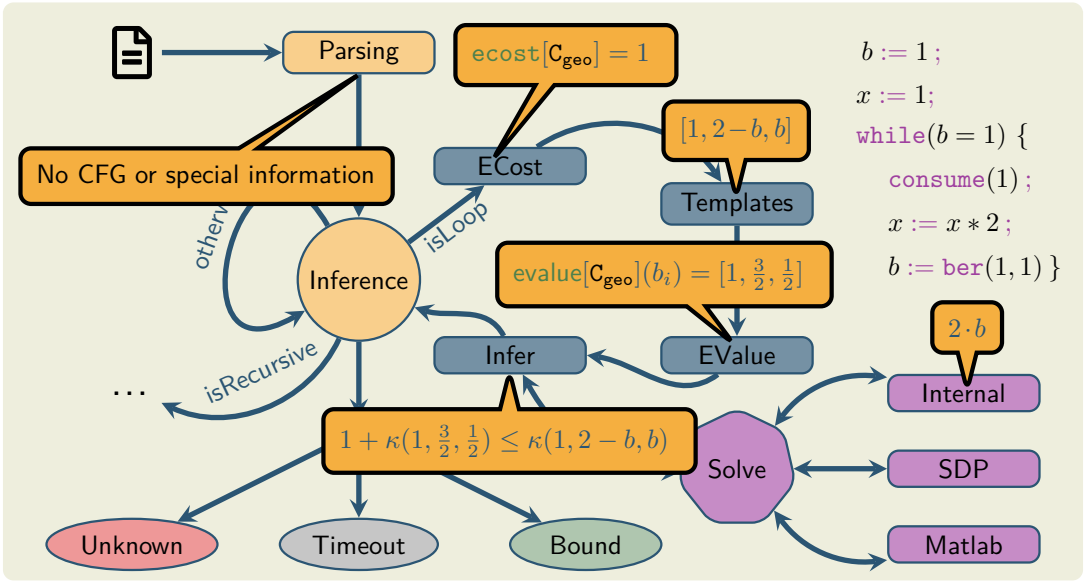
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1 + κ(1, 3/2, 1/2) ≤ κ(1, 2 − b, b)
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2·b
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SDP
Matlab
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Timeout
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b := 1 ;
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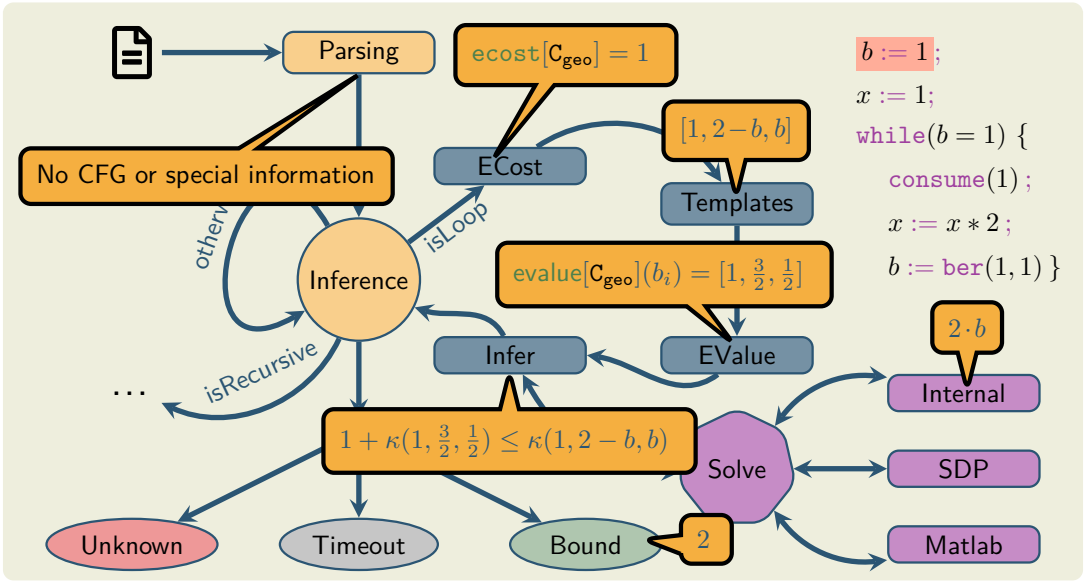
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2·b
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b := ber(1, 1) }

Summary

- static program analysis for probabilistic programs

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- ERT calculus for compositional analysis
- ECT calculus for compositional/modular analysis

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Current/Future Research

- finishing recursion (modularity?)
- SDP solving

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Current/Future Research

- finishing recursion (modularity?)
- SDP solving
- logarithmic bounds
- abstractions via coupling

Thank you for your attention!

References

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