universität innsbruck

Abstraction, Refinement and Proof in a Probabilistic Setting

Master Seminar WS20/21

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November 25, 2020 & December 1, 2020

Motivation

- model natural/physical processes $\Rightarrow$ "real" coin flip

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- primality tests $\Rightarrow$ cryptography
- machine learning
- improvement of algorithms, e.g., quicksort

Motivation - Quicksort

- "standard" vs. randomized quicksort

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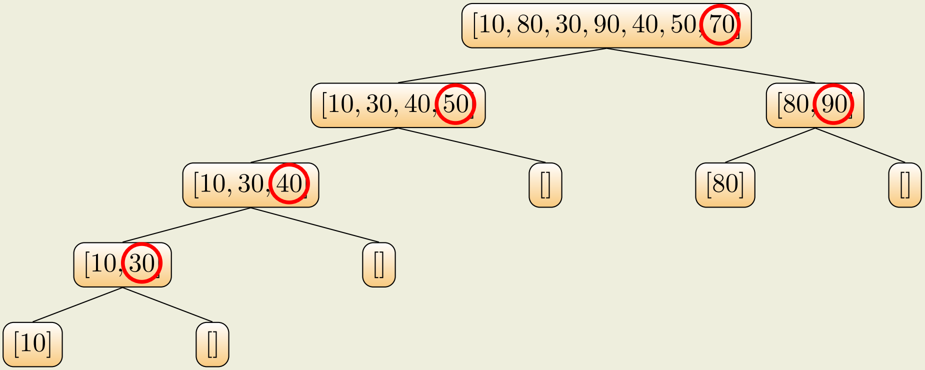
Example Quicksort



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- *first* vs. *last* vs. *random* vs. *median* pivot element

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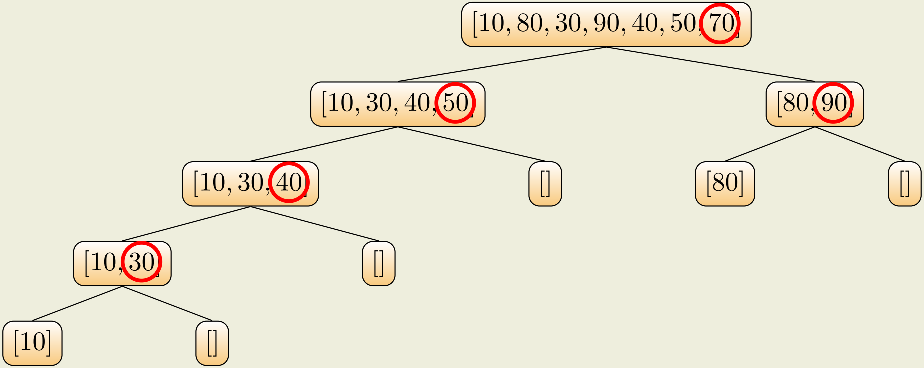

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- "standard" vs. randomized quicksort
- *first* vs. *last* vs. *random* vs. *median* pivot element
- worst case: $\mathcal{O}(n^2)$ vs. $\mathcal{O}(n^2)$ (BUT expected or average time complexity is $\mathcal{O}(n \log n)$)

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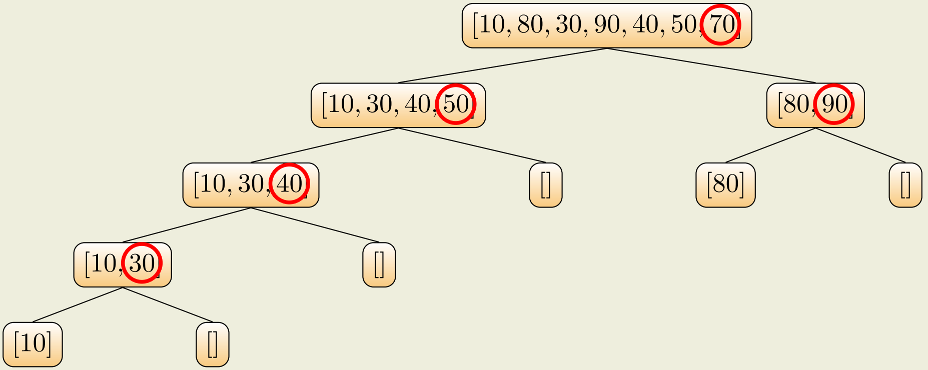

Overview

- Guarded Command Language (GCL)
- Probabilistic Guarded Command Language (pGCL)
- Abstraction and Refinement
- Probably Hoare? Hoare Probably!
- Abstraction Refinement and Proof for Probabilistic Systems

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Syntax of GCL

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\begin{aligned}
\langle \text{guarded command} \rangle &::= \langle \text{guard} \rangle \rightarrow \langle \text{guarded list} \rangle \\
\langle \text{guard} \rangle &::= \langle \text{boolean expression} \rangle \\
\langle \text{guarded list} \rangle &::= \langle \text{statement} \rangle \{ \mathbf{;} \langle \text{statement} \rangle \} \\
\langle \text{guarded command set} \rangle &::= \langle \text{guarded command} \rangle \{ \Box \langle \text{guarded command} \rangle \}
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Alternative Construct

$$\begin{aligned}&\textbf{if } x \geq y \rightarrow m := x\\&\square\ y \geq x \rightarrow m := y\\&\textbf{fi}\end{aligned}$$

Alternative Construct (Nondeterminism)

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Repetitive Construct

$$
\begin{array}{ll}
k := 0; j := 1; & \\
\textbf{do } j \neq n \to & \textbf{if } \mathsf{f}(j) \leq \mathsf{f}(k) \to j := j + 1 \\
 & \Box \; \mathsf{f}(j) \geq \mathsf{f}(k) \to k := j; j := j + 1 \\
 & \textbf{fi} \\
\textbf{od} &
\end{array}
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Primer: Nondeterminism vs. Determinism

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Determinism Example DE

$$
\begin{aligned}
&\textbf{if } x > y \rightarrow m := x \\
&\square\ y < x \rightarrow m := y \\
&\square\ y = x \rightarrow m := y \\
&\textbf{fi}
\end{aligned}
$$

Primer cont'd

"Assertions about programs" are predicates that are supposed to be "true at this point of the program".

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- reasoning about weakest pre-conditions of programs ⇒ weakest pre-condition semantics
- Hoare logic = formal system (set of logical rules) for reasoning about the correctness of programs

How to Use GCL in a Probabilistic Setting?

- *deterministic* vs. *nondeterministic* vs. *probabilistic* choice
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Demonic Choice Operator

$$\mathit{this} \sqcap \mathit{that}$$

Basically means, that it does not matter if we choose *this* or *that*.

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Probabilistic Choice Operator: Coin Flip

Tail ${}_{\frac{1}{2}}\!\oplus$ *Head* . . . fair coin

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no perfect coins in nature:

$$\textit{Tail}\ {}_{0.49}\!\oplus \textit{Head} \sqcap \textit{Tail}\ {}_{0.51}\!\oplus \textit{Head} \qquad \ldots \text{nearly fair coin}$$

pGCL cont'd

There exist more constructs such as:

- Boolean embedding of predicate *pred* as expectation:

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pGCL cont'd

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- Multi-way probabilistic choices
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- Demonic choice in variable assignments
- Iteration

$$\textbf{do}\ \textit{pred} \rightarrow \textit{body}\ \textbf{od} := (\textbf{mu}\ xxx \cdot (\textit{body}; xxx)\ \textbf{if}\ \textit{pred}\ \textbf{else skip})$$

- Implication-like relations for expectations *exp*, *exp'*:

$exp \Rrightarrow exp'$	means	*exp* is everywhere less than or equal to *exp'*
$exp \equiv exp'$	means	*exp* and *exp'* are everywhere equal
$exp \Lleftarrow exp'$	means	*exp* is everywhere greater than or equal to *exp'*

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\textit{wp}.(\textit{prog} \,{}_p\!\oplus\, \textit{prog'}).\textit{postE} &:= p * \textit{wp}.\textit{prog}.\textit{postE} + (1-p) * \textit{wp}.\textit{prog'}.\textit{postE}
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What is Abstraction?

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What is Refinement? (Specialization)

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Consider the input set $\mathcal{I}$ for functions/programs f, g, then g is a refinement of f if

$$\{g(i) \mid i \in \mathcal{I}\} \subset^* \{f(i) \mid i \in \mathcal{I}\}$$

*: N.B.: This is not true for all types of abstraction or how abstraction is used.

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\equiv\; & \frac{[y < 0]}{3} + [y = 0] + \frac{2[+y \geq 0]}{3}
\end{aligned}$$

Example cont'd

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How can we build a more abstract program of this Example?

$$x := -y \;{}_{\frac{1}{3}}\!\oplus\; x := +y$$

Example Abstraction

- $x := -y$ is to be executed with probability at least $\frac{1}{3}$
- $x := +y$ is to be executed with probability at least $\frac{1}{4}$
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What else can we say from this specification?

$$x := -y \;{}_{\frac{1}{3}}\!\oplus x := +y \sqcap x := -y \;{}_{\frac{3}{4}}\!\oplus x := +y$$

We can also specify that a program part is executed given some range of probability.

Example Abstraction cont'd

$$(x := -y \ {}_{\frac{1}{3}}\!\oplus x := +y) \sqcap (x := -y \ {}_{\frac{3}{4}}\!\oplus x := +y)$$

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Using again the *wp*-semantics, we compute the following

$$wp.((x := -y \ {}_{\frac{1}{3}}\!\oplus x := +y) \sqcap (x := -y \ {}_{\frac{3}{4}}\!\oplus x := +y)).[x \geq 0]$$

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Example Refinement

Refinement is the converse of abstraction:

$$S \sqsubseteq T := \textit{wp}.S.R \Rrightarrow \textit{wp}.T.R \qquad \text{for all } R$$

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This programs is a refinement according to the specification:

$$x := -y \;{}_{\frac{1}{2}}\!\oplus x := +y$$

Example Refinement cont'd

Prove the following refinement relation:

$$(x := -y \ {}_{\frac{1}{3}}\!\oplus\ x := +y) \sqcap (x := -y \ {}_{\frac{3}{4}}\!\oplus\ x := +y)$$
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\Rrightarrow\; & x := -y \;{}_{\frac{1}{2}}\!\oplus\; x := +y
\end{aligned}
$$

Semantic Level

$$
\mathit{wp}.(x := -y \;{}_{\frac{1}{2}}\!\oplus\; x := +y).P
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$$\mathit{wp}.(x := -y \ {}_{\frac{1}{2}}\!\oplus x := +y).P$$
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because $\frac{3}{5} * \frac{1}{3} + \frac{2}{5} * \frac{3}{4} = \frac{1}{2}$
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N.B.: Demonic choice can be refined by any probabilistic choice.

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The probability that two fair coins, when flipped, show the same faces:

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For our overall reasoning we only need the second interpretation and the first one is only "glue" that holds our reasoning together.

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All GCL commands satisfy conjunctivity:

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We do not have conjunctivity in pGCL, but sub-linearity (it generalizes conjunctivity): Let a, b, c be non-negative finite reals, and P, P' expectations, then all pGCL constructs satisfy

$$\mathit{wp}.S.(aP + bP' \ominus c) \Leftarrow a(\mathit{wp}.S.P) + b(\mathit{wp}.S.P') \ominus c$$

Properties of pGCL

All GCL commands satisfy conjunctivity:

$$\textit{wp}.S.(P \wedge P') = \textit{wp}.S.P \wedge \textit{wp}.S.P'$$

Do we need that also for pGCL?

We do not have conjunctivity in pGCL, but sub-linearity (it generalizes conjunctivity): Let a, b, c be non-negative finite reals, and P, P' expectations, then all pGCL constructs satisfy

$$\textit{wp}.S.(aP + bP' \ominus c) \Lleftarrow a(\textit{wp}.S.P) + b(\textit{wp}.S.P') \ominus c$$

where truncated subtraction $\ominus$ is defined as $x \ominus y := (x - y) \max 0$

Monotonicity

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Scaling

Multiplication by a non-negative constant distributes through commands. Note we already have one direction due to sub-linearity:

$$c * \mathit{wp}.S.P \Rrightarrow \mathit{wp}.S.(c * P)$$

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We define $\&$ as:

$$exp \;\&\; exp' := exp + exp' \ominus 1$$

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Sub-Conjunctivity

As $\&$ sub-distributes through expectation transformers and from sub-linearity with $a, b, c := 1, 1, 1$ we have:

$$\mathit{wp}.S.P \;\&\; \mathit{wp}.S.P' \Rrightarrow \mathit{wp}.S.(P \;\&\; P')$$

for all S.

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The Programs Cannot Be Distinguished

all post-conditions	*fair* probabilities		*unfair* probabilities
false	0	$= 0 =$	0
$x = A$	$1 \min 0$	$= 0 =$	$\frac{1}{2}(1 \min 0) + \frac{1}{2}(1 \min 0)$
$x = B$	$0 \min \frac{1}{2}$	$= 0 =$	$\frac{1}{2}(0 \min 1) + \frac{1}{2}(0 \min 0)$
$x = C$	$0 \min \frac{1}{2}$	$= 0 =$	$\frac{1}{2}(0 \min 0) + \frac{1}{2}(0 \min 1)$
$x \neq A$	$0 \min 1$	$= 0 =$	$\frac{1}{2}(0 \min 1) + \frac{1}{2}(0 \min 1)$
$x \neq B$	$1 \min \frac{1}{2}$	$= \frac{1}{2} =$	$\frac{1}{2}(1 \min 0) + \frac{1}{2}(1 \min 1)$
$x \neq C$	$1 \min \frac{1}{2}$	$= \frac{1}{2} =$	$\frac{1}{2}(1 \min 1) + \frac{1}{2}(1 \min 0)$
true	1	$= 1 =$	1

Hoare Probably!

Let $preExp$, $postExp$ be real-valued expressions in the program variables:

$$\{preExp\}\ prog\ \{postExp\}$$

$preExp$ evaluated in the initial state gives a lower bound for the expected value of expression $postExp$.

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$postExp$	*fair* $preExp$	*unfair* $preExp$
$[x = A] + 2[x = B]$	1	$\frac{1}{2}$

fair* refines *unfair

Sub-Linearity

For any reals $a, b, c \geq 0$ and expectations $preExp$, $preExp'$, $postExp$, $postExp'$, from

$$\{preExp\}\ S\ \{postExp\}$$
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Monte Carlo Algorithms

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$$b := B \;{}_{\geq p}\!\oplus\; (b := True \sqcap b := False)$$

where B is the desired result.

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where B is the desired result.
It is equal to:

$$b := B \sqcap \left(b := B \;{}_{p}\!\oplus\; (b := True \sqcap b := False)\right)$$

Probabilistic Primality

Instantiation with probabilistic primality:

if B **then** $b := True$ **else**

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Furthermore we need an invariant and choose:

$$Inv = [b] \lhd B \rhd 1 - \frac{[b]}{2^n}$$

Probabilistic Primality cont'd

$$
\begin{array}{l}
b, n := True, N; \\
\textbf{do } n \neq 0 \wedge b \rightarrow \\
\qquad CheckOnce; \\
\qquad n := n - 1 \\
\textbf{od}
\end{array}
$$

Probabilistic Primality cont'd

$$\{1 \lhd B \rhd 1 - \frac{1}{2^N}\}$$

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$$1 \lhd B \rhd 1 - \frac{1}{2^N} \Rrightarrow Inv[b, n := True, N]$$
$$= 1 \lhd B \rhd 1 - \frac{1}{2^N} \Rrightarrow [True] \lhd B \rhd 1 - \frac{[True]}{2^N}$$

- Check for post-condition on termination:
  When $n = 0$ or $\neg b$ holds:

$$1 \lhd B \rhd 1 - \frac{1}{2^N} \Rrightarrow [b] \lhd B \rhd [\neg b]$$

Probabilistic Primality cont'd

- Loop body including decrement in invariant:

$$\{1 \lhd B \rhd 1 - \frac{1}{2^n}\}\ CheckOnce\ \{1 \lhd B \rhd 1 - \frac{1}{2^{n-1}}\}$$

and we assume the truth of the loop guard $\Rightarrow n \neq 0 \wedge b$.

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$$\textbf{if } B \textbf{ then } b := True \textbf{ else} \cdots$$

  + When B does not hold, we use $b := False$, which makes both expectations 1.
    $\Rightarrow$ Which is also part of our instantiation.

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The other part is

$$b := False \ {}_{\frac{1}{2}}\!\oplus\ b := True$$

. For this we have an inference rule for probabilistic choice:

Probabilistic Primality cont'd

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$$\frac{\begin{array}{lll}\{preExp\} & S & \{postExp\} \\ \{preExp'\} & S' & \{postExp\}\end{array}}{\{p \times preExp + (1-p) \times preExp'\}\ S \;{}_{p}\!\oplus\; S'\ \{postExp\}}$$

Probabilistic Primality cont'd

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$$b := False \;{}_{\frac{1}{2}}\!\oplus\; b := True$$

. For this we have an inference rule for probabilistic choice:

$$\frac{\begin{array}{lr} \{1\}\; b := False & \{1 - \frac{[b]}{2^{n-1}}\} \\ \{1 - \frac{1}{2^{n-1}}\}\; b := True & \{1 - \frac{[b]}{2^{n-1}}\} \end{array}}{\{\frac{1}{2}1 + \frac{1}{2}(1 - \frac{1}{2^{n-1}})\}\; b := False \;{}_{\frac{1}{2}}\!\oplus\; b := True\; \{1 - \frac{[b]}{2^{n-1}}\}}$$

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We just need to calculate the pre-expectation:

$$\frac{1}{2}1 + \frac{1}{2}(1 - \frac{1}{2^{n-1}}) = 1 - \frac{1}{2^n} = \text{“note that b holds”}\ 1 - \frac{[b]}{2^n}$$

Thus $CheckOnce$ is implemented by our instantiation.

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- Guarded Command Language (GCL)
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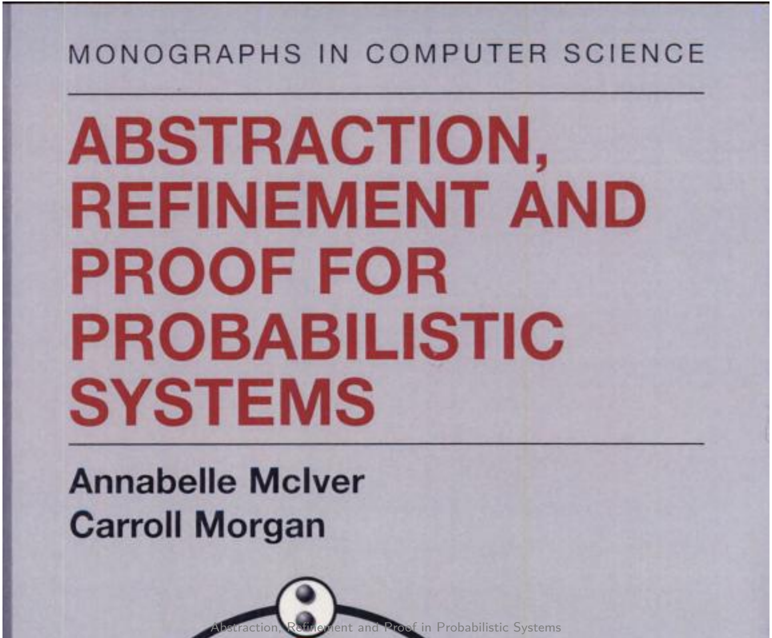
Abstraction Refinement and Proof for Probabilistic Systems



Short Overview of the Book

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- Part II: Semantic structures: this part develops in detail the mathematics on which the probabilistic logic is built and with which it is justified (correctness).
- Part III: Advanced topics: this part concentrates on more exotic methods of specification and design, in this case probabilistic temporal/modal logics.
- Part IV: Appendices, bibliography and indexes

Summary

- GCL $\Rightarrow$ pGCL
- *wp*-semantics of pGCL
- healthiness properties of pGCL
- probably Hoare semantics vs. Hoare probably semantics

Thank you for your attention!

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