universität innsbruck

Expected Cost Analysis of Probabilistic Programs

Lunchtime Seminar

Martin Avanzini Georg Moser Michael Schaper Jonas Schöpf

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Probabilistic Programming

Probabilistic Programming is a programming paradigm where probabilistic models can be specified and inference for these is done automatically. Languages in this class, e.g., incorporate random events as primitives or probabilistic branching.

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- cryptography $\Rightarrow$ primality tests
- robotics/machine learning algorithms

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- expressivity to model unavoidable application specifics (i.e. fault tolerance)
- cryptography ⇒ primality tests
- robotics/machine learning algorithms
- improvement of algorithms, e.g., quicksort

Motivation - Quicksort

- "standard" vs. randomized quicksort

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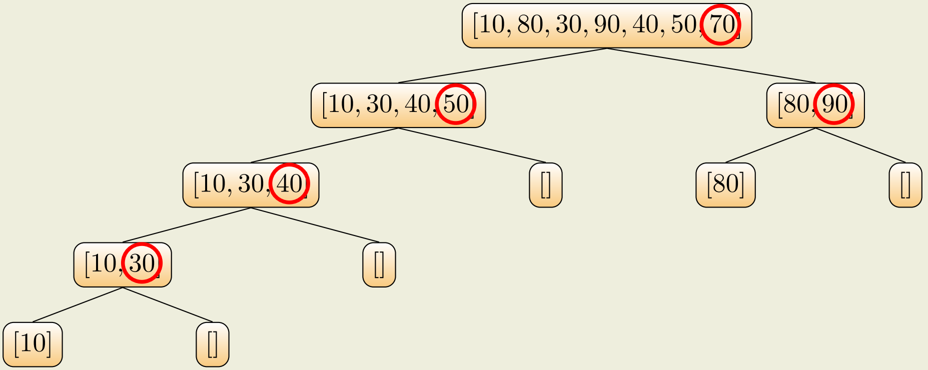
Example Quicksort



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- *first* vs. *last* vs. *random* vs. *median* pivot element

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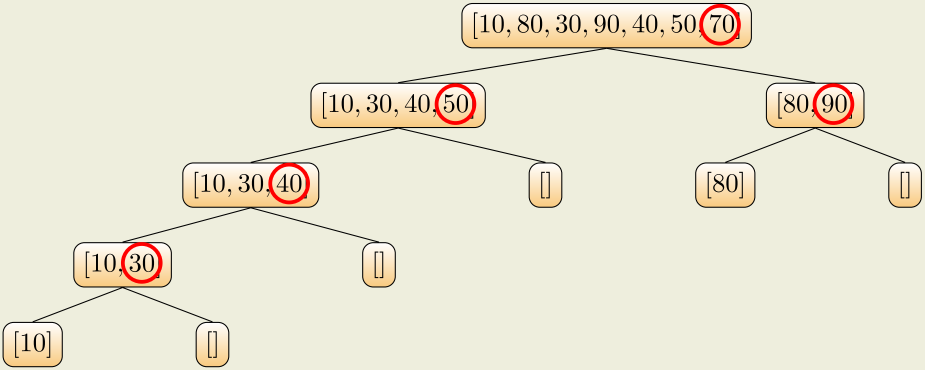
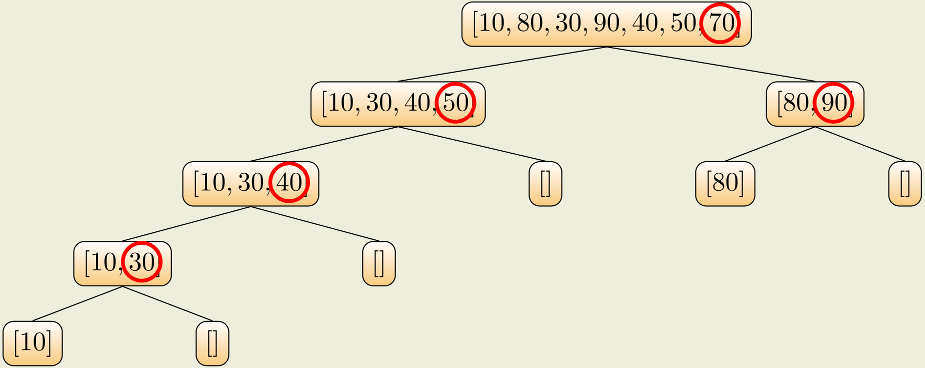


Motivation - Quicksort

- "standard" vs. randomized quicksort
- *first* vs. *last* vs. *random* vs. *median* pivot element
- worst case: $\mathcal{O}(n^2)$ vs. $\mathcal{O}(n^2)$ (BUT expected or average time complexity is $\mathcal{O}(n \log n)$)

Example Quicksort



Overview

- Primer
- Syntax & Semantic
- Automation
- Constraint Solving
- Summary

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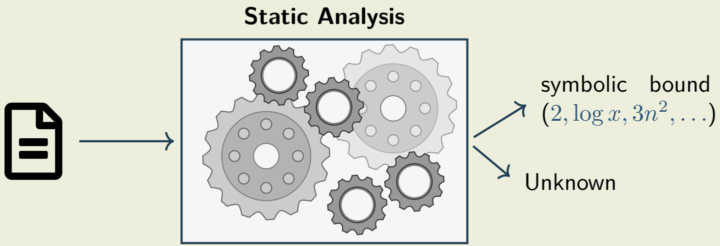
Static Resource Analysis



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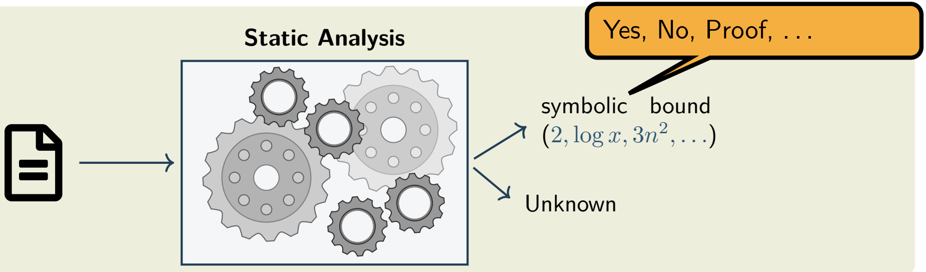

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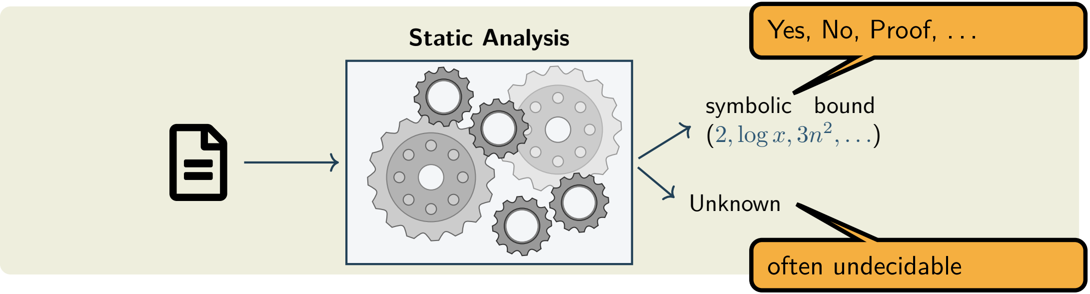

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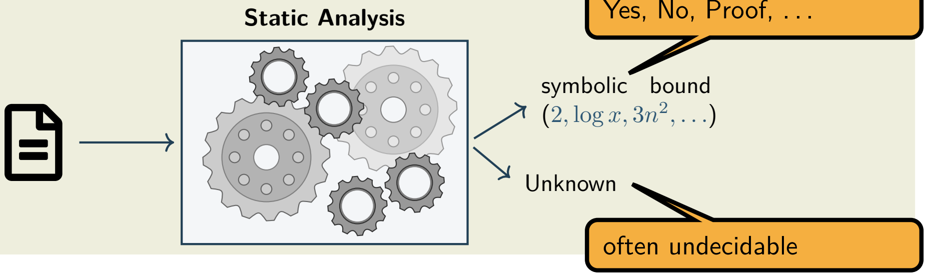
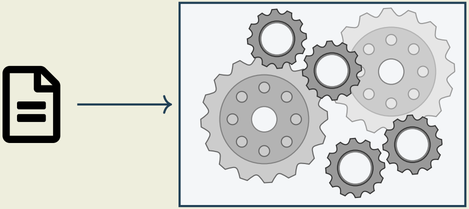


- integral part of formal verification
- improving the quality of complex software
- medical software, aviation software, nuclear software, ...

Static Resource Analysis

Static Analysis



symbolic bound $(2, \log x, 3n^2, \ldots)$

Yes, No, Proof, ...

Unknown

often undecidable

- integral part of formal verification
- improving the quality of complex software
- medical software, aviation software, nuclear software, ...

- recurrence relations
- type systems
- term rewriting
- ...

Non-/Deterministic vs. Probabilistic

Non-/Determi. | Probabilistic

Dynamics

Semantics

Non-/Deterministic vs. Probabilistic

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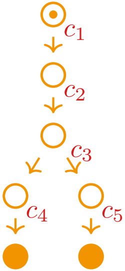
Dynamics



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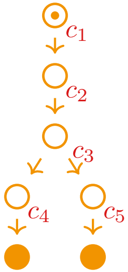
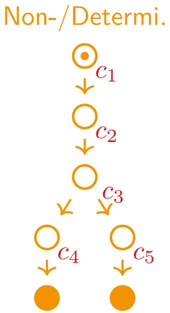
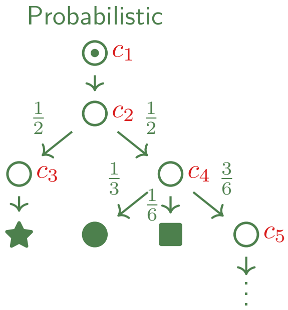


Semantics

- assign cost c_i to each operation
- overall cost is the sum of all operation costs

Non-/Deterministic vs. Probabilistic





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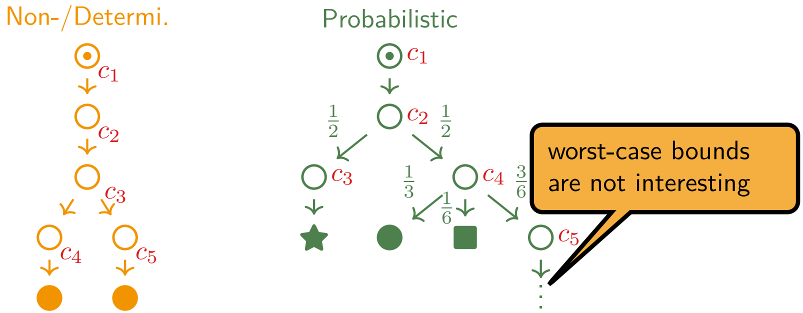

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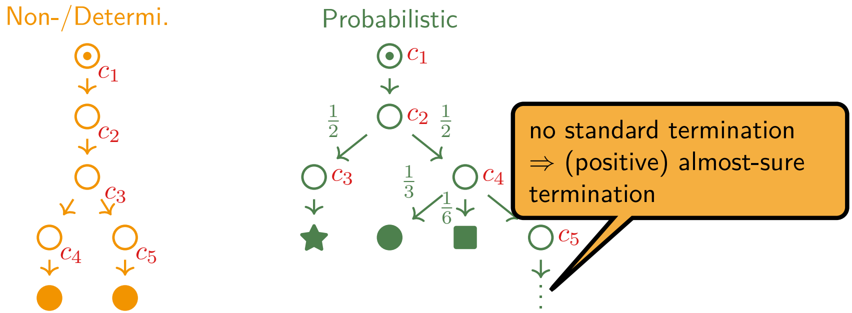

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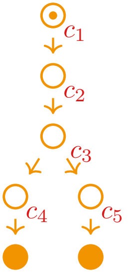

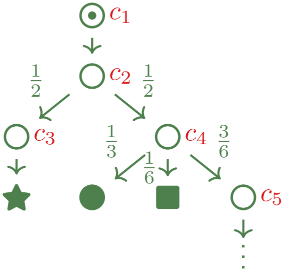

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$[\![P]\!](\odot) = \bullet$

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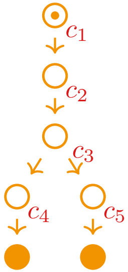
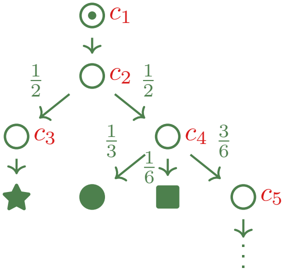
Dynamics		
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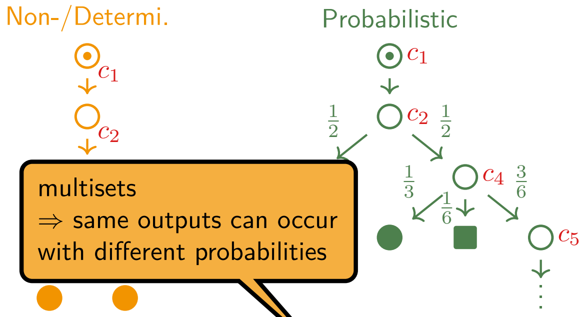

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What Do We Want to Achieve?

We would like to have a calculus which to determine the expected runtime of a probabilistic program or algorithm.

- compositional
- modular
- precise

Furthermore it would be beneficial if termination follows from this calculus.

Probabilistic While (pWhile)

- inspired by Dijkstra's Guarded Command Language (GCL)

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Syntax of pWhile

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\begin{aligned}
\mathtt{C}, \mathtt{D} ::= \\
&\mid \mathtt{skip} \\
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&\mid \mathtt{C};\mathtt{D} \\
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$\texttt{rand}(e), \texttt{unif}(n, m), \texttt{ber}(n, m), \ldots$

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$\mathtt{prob}(n, m)\ \{\mathtt{C}\}\ \{\mathtt{D}\}$

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Example – geo

```
b := 1; x := 1;
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   b := ber(1, 1)}
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\mathsf{ect}[\mathsf{geo}](0) &= 1 + \frac{1}{2} \cdot (1 + \frac{1}{2} \cdot (1 + \ldots \\
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\end{aligned}
$$

Experiments Stable Version ecoimp

Problem	ecoimp	Absynth	Wang et al. 2019
linear			
2drwalk	0.026	0.286	−−
bayesian_network	0.002	0.127	−−
ber	0.001	0.014	6.684
C4B_t13	0.005	0.025	8.527
non-linear			
2drobot	1.760	−−	11.621
queueing-network	2.215	1.286	78.191
nest-4	0.554	−−	−−
trader-10	0.025	3.638	10.460
trader-20	0.030	119.464	10.420
trader-100000	2.113	−−	20.332
coupons-n	0.195	−−	−−

Expected Cost Transformer

We define the expected cost transformer (ECT) operating on cost functions over states. Thus $\mathsf{ect}[\mathtt{C}](f)$ can be seen as the cost of $\mathtt{C}$ w.r.t. a continuation cost f.

C	$\mathsf{ect}[\mathtt{C}](f)$	$\mathsf{evalue}[\mathtt{C}](f)$
$\mathtt{consume}(e)$	$\langle e\rangle + f$	f
$\mathtt{skip}$	f	f
$\mathtt{abort}$	0	0
$x := d$	$\lambda\sigma.\mathbb{E}_{d(\sigma)}(\lambda v.f[x/v](\sigma))$	$\lambda\sigma.\mathbb{E}_{d(\sigma)}(\lambda v.f[x/v](\sigma))$
$\mathtt{C};\mathtt{D}$	$\mathsf{ect}[\mathtt{C}](\mathsf{ect}[\mathtt{D}](f))$	$\mathsf{evalue}[\mathtt{D}](\mathsf{evalue}[\mathtt{D}](f))$
$\mathtt{if}(\phi)\ \{\mathtt{C}\}\ \{\mathtt{D}\}$	$[\phi]\cdot\mathsf{ect}[\mathtt{C}](f) + [\neg\phi]\cdot\mathsf{ect}[\mathtt{D}](f)$	$[\phi]\cdot\mathsf{evalue}[\mathtt{C}](f) + [\neg\phi]\cdot\mathsf{evalue}[\mathtt{D}](f)$
$\{\mathtt{C}\} <> \{\mathtt{D}\}$	$\max(\mathsf{ect}[\mathtt{C}](f), \mathsf{ect}[\mathtt{D}](f))$	$\max(\mathsf{evalue}[\mathtt{C}](f), \mathsf{evalue}[\mathtt{D}](f))$
$\mathtt{while}(\phi)\ \{\mathtt{C}\}$	$\mathtt{lfp}(\lambda F.[\phi]\cdot\mathsf{ect}[\mathtt{C}](F) + [\neg\phi]\cdot f)$	$\mathtt{lfp}(\lambda F.[\phi]\cdot\mathsf{evalue}[\mathtt{C}](F) + [\neg\phi]\cdot f)$

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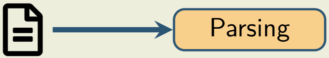

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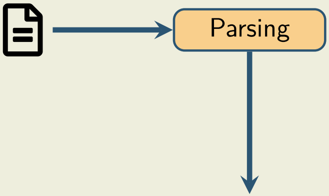

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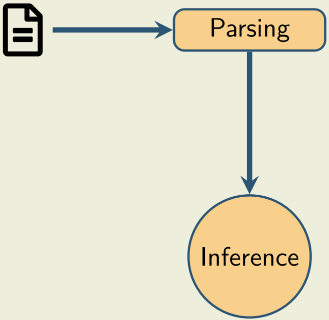

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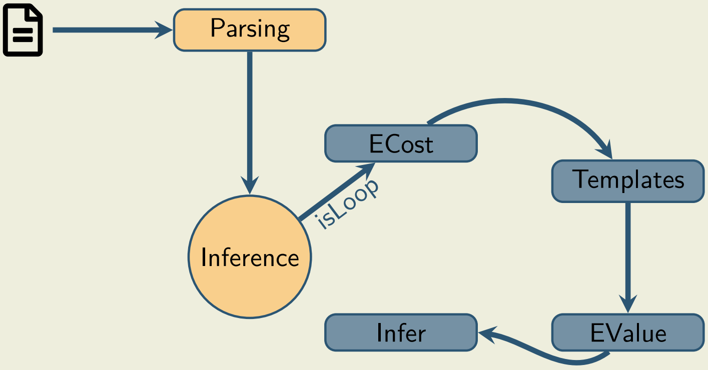

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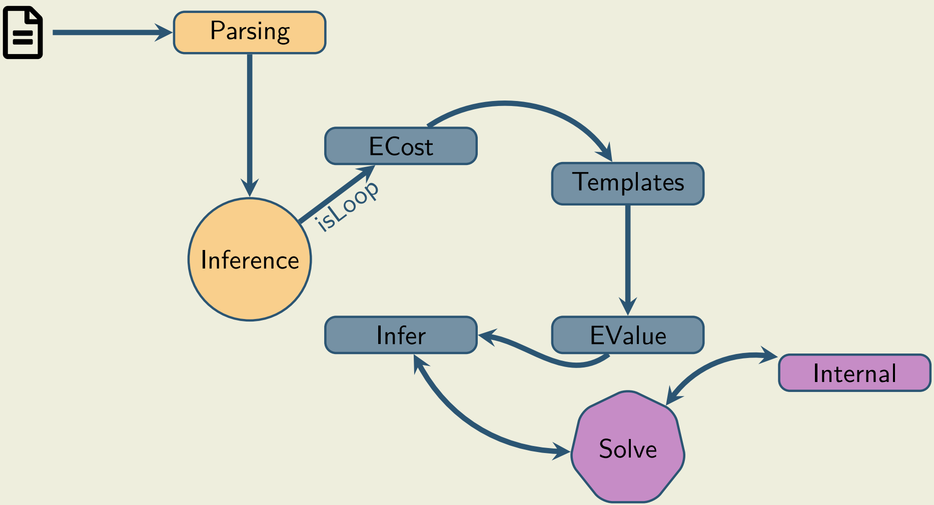
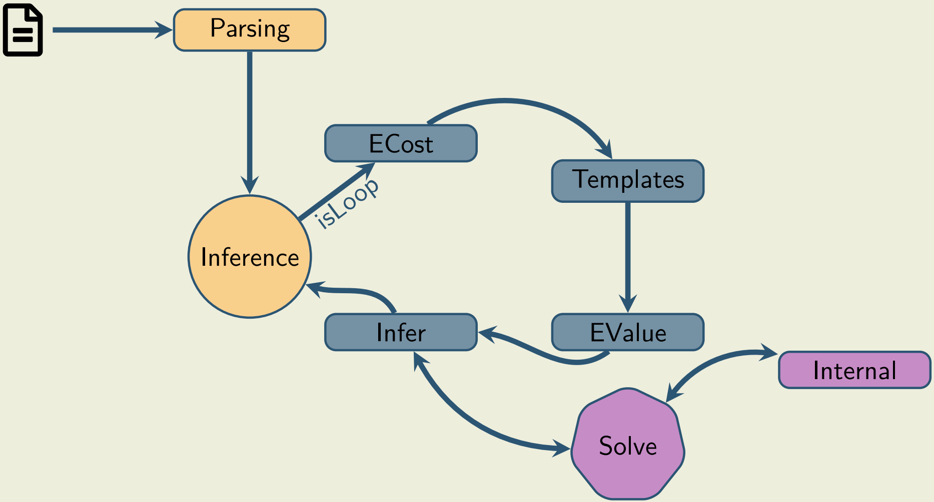


Parsing
Inference
isLoop
ECost
Templates
EValue
Infer
Internal
Solve

ecoimp



Parsing
ECost
Templates
isLoop
Inference
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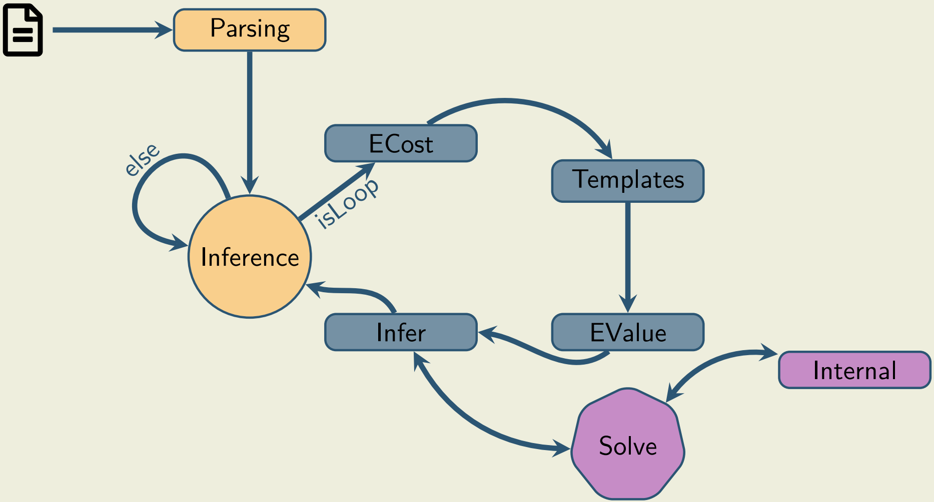
ecoimp



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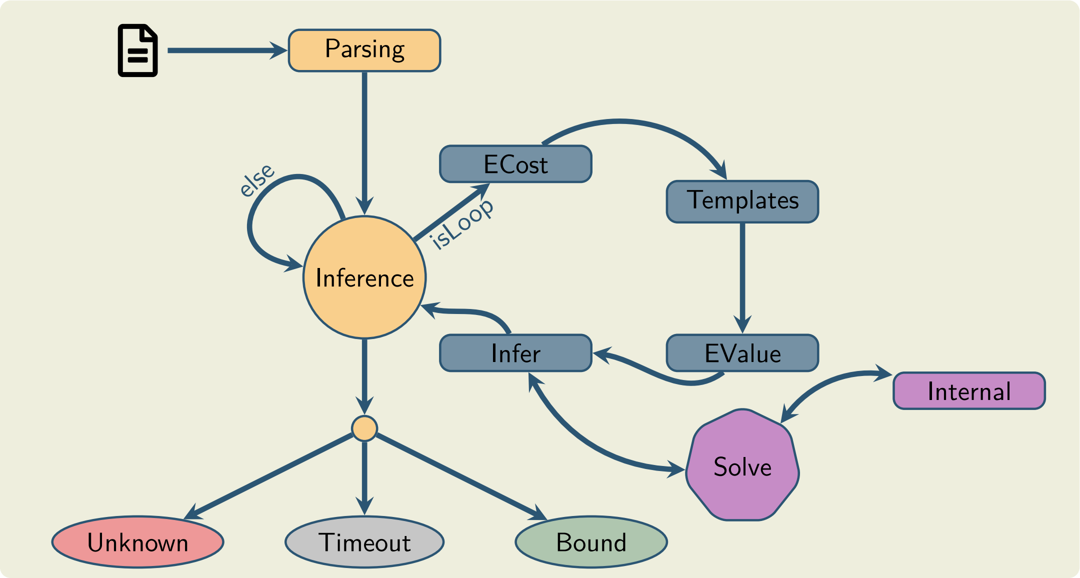

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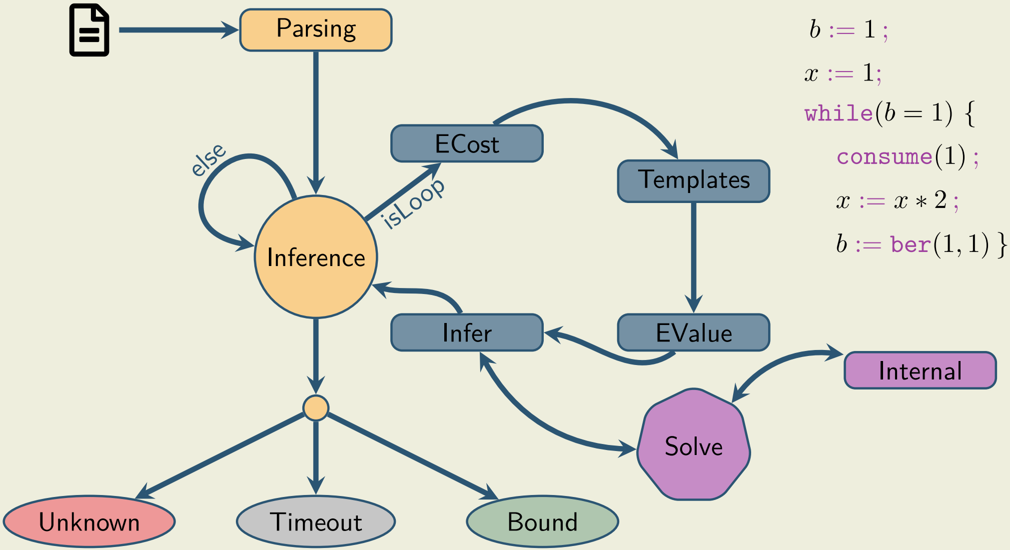

$b := 1\,;$

$x := 1;$

$\texttt{while}(b = 1)\ \{$

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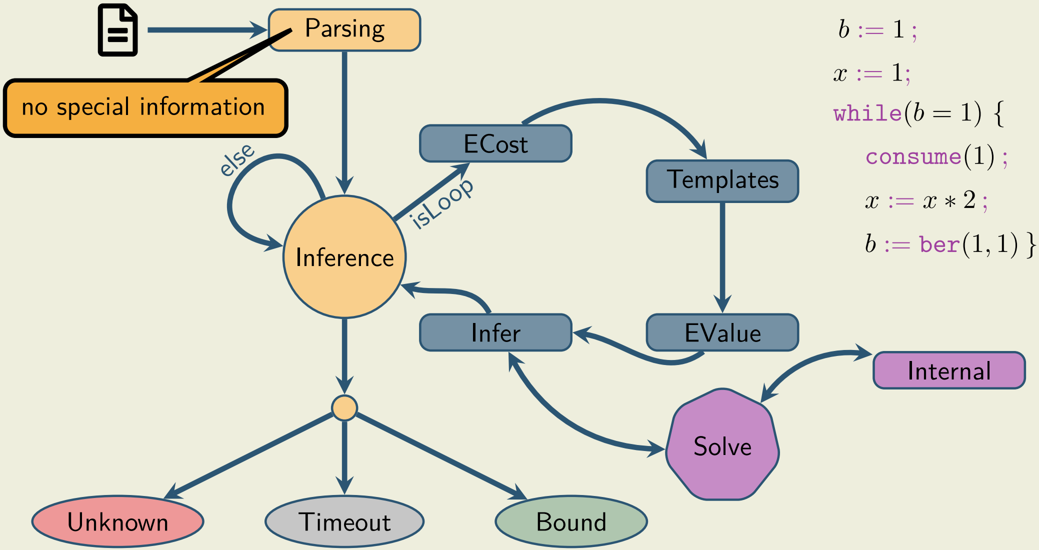
$\quad b := \texttt{ber}(1, 1)\ \}$

ecoimp



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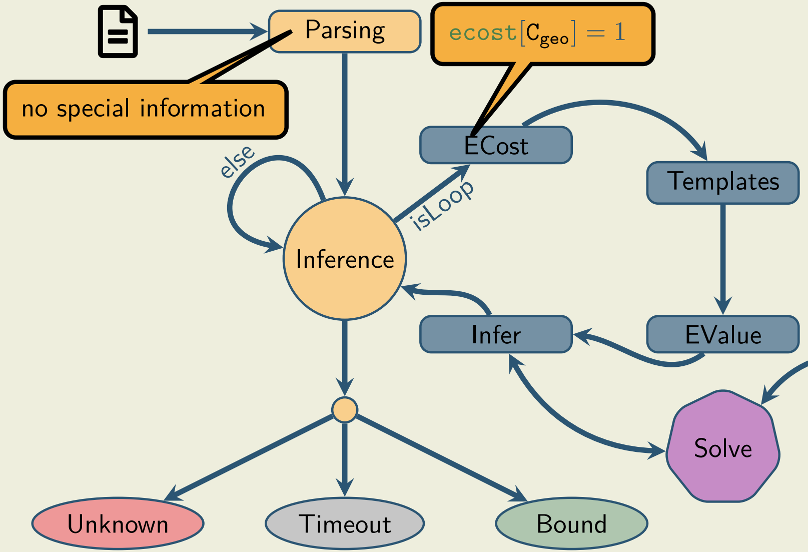

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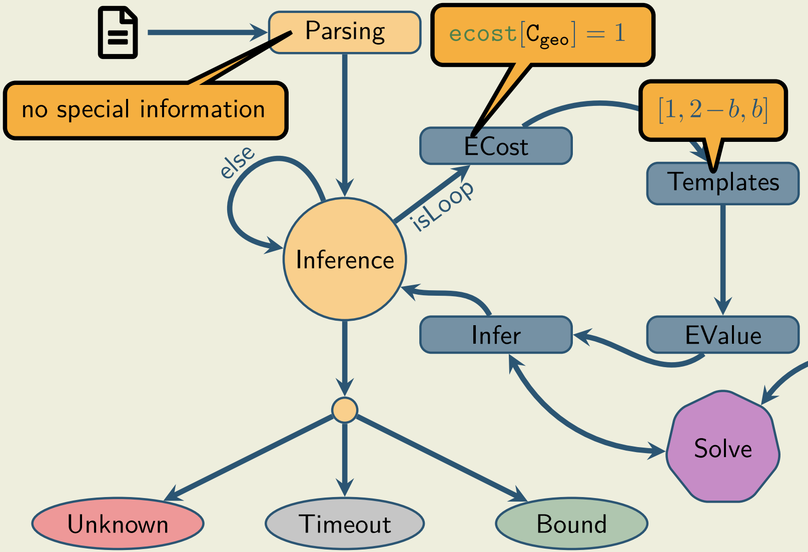

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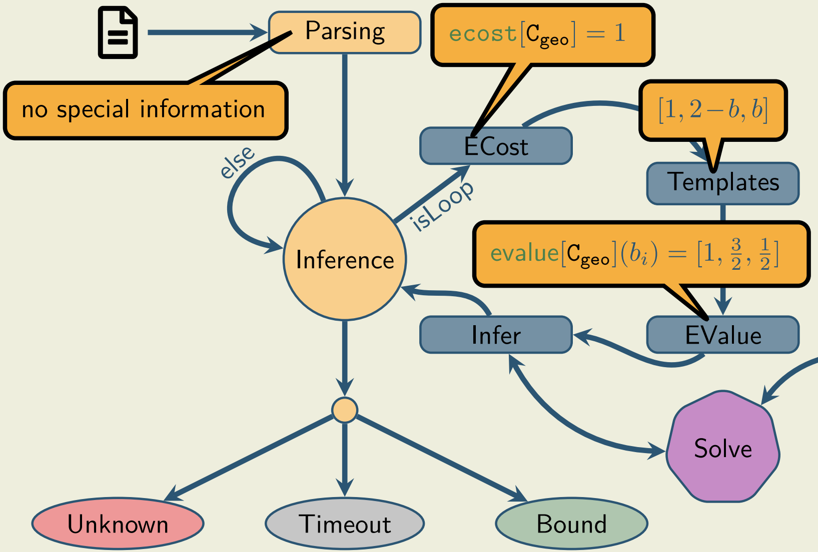

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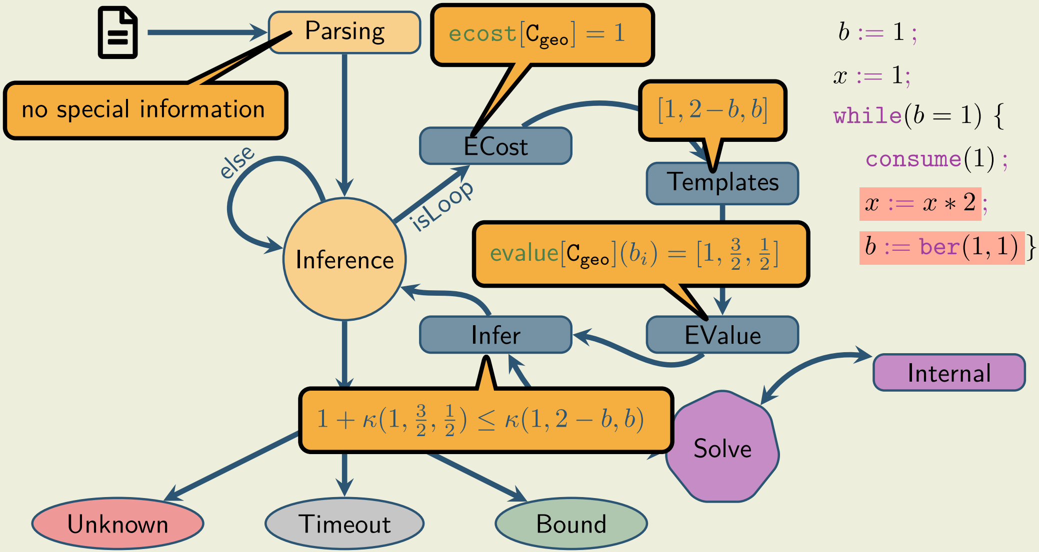


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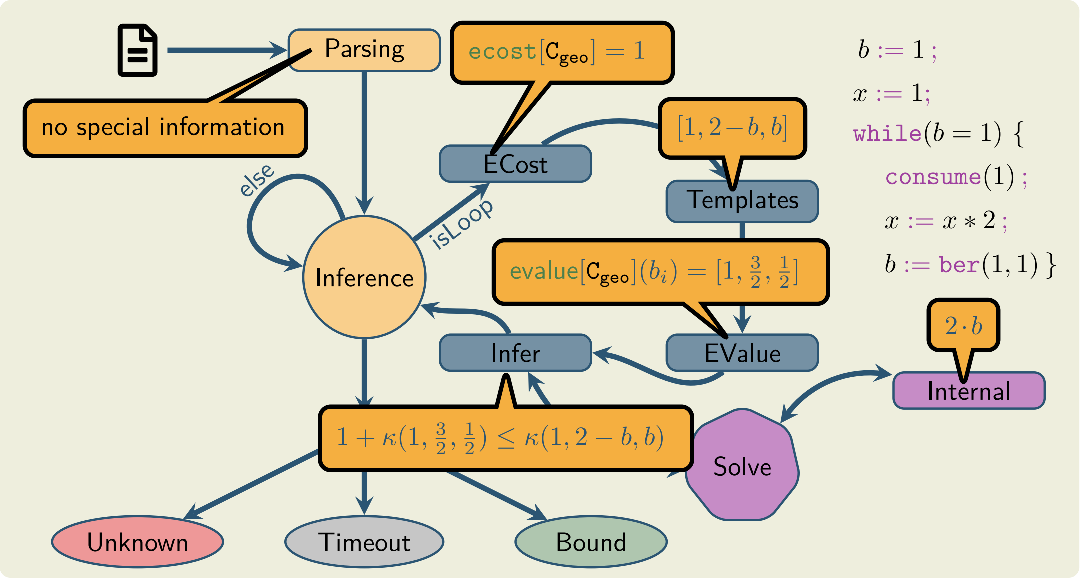
ecoimp


Parsing
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ECost
$[1, 2-b, b]$
Templates
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Inference
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Infer
EValue
$2 \cdot b$
Internal
$1 + \kappa(1, \frac{3}{2}, \frac{1}{2}) \leq \kappa(1, 2-b, b)$
Solve
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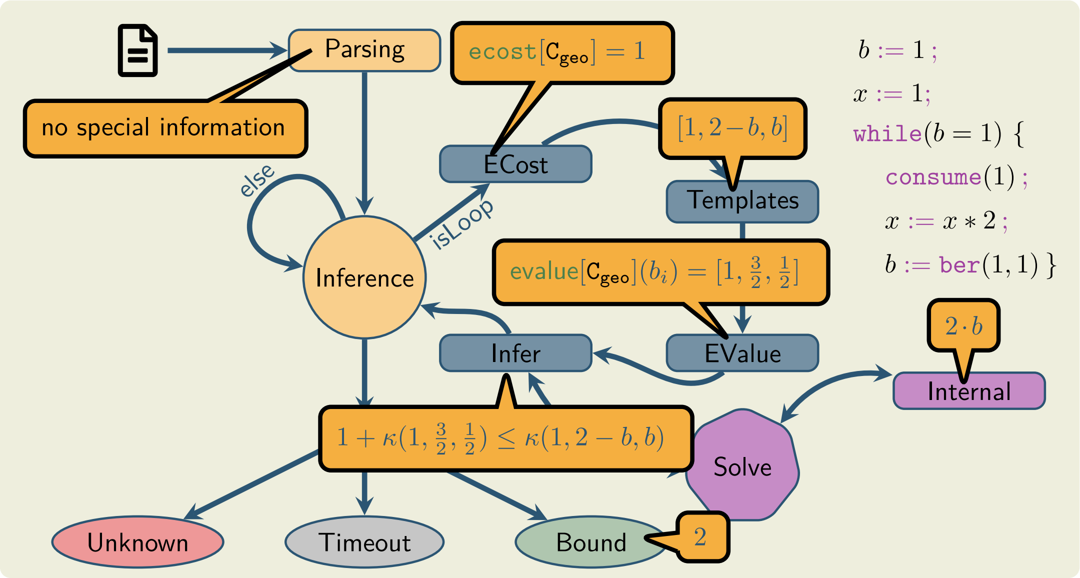

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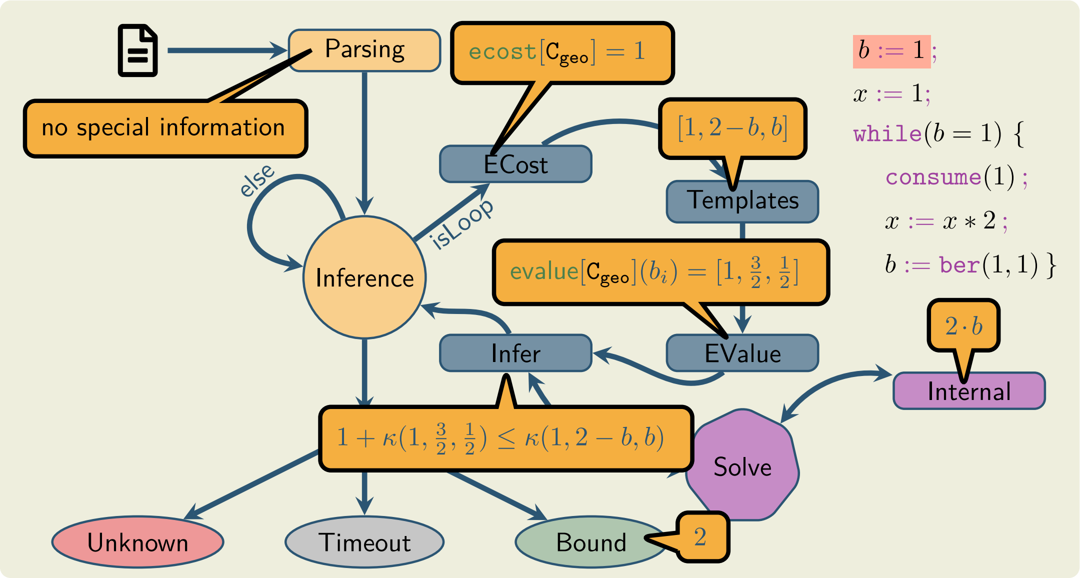
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Timeout
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$b := 1$;
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$x := x * 2$;
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Recursion

We extend the syntax of `pWhile` to function definitions and a call statement to a function. A program is now a sequence of functions.

$$\mathtt{F} ::= \mathtt{def}\ fun\ \mathtt{:}\ \{\mathtt{C}\} \qquad\qquad \mathtt{call}\ fun$$

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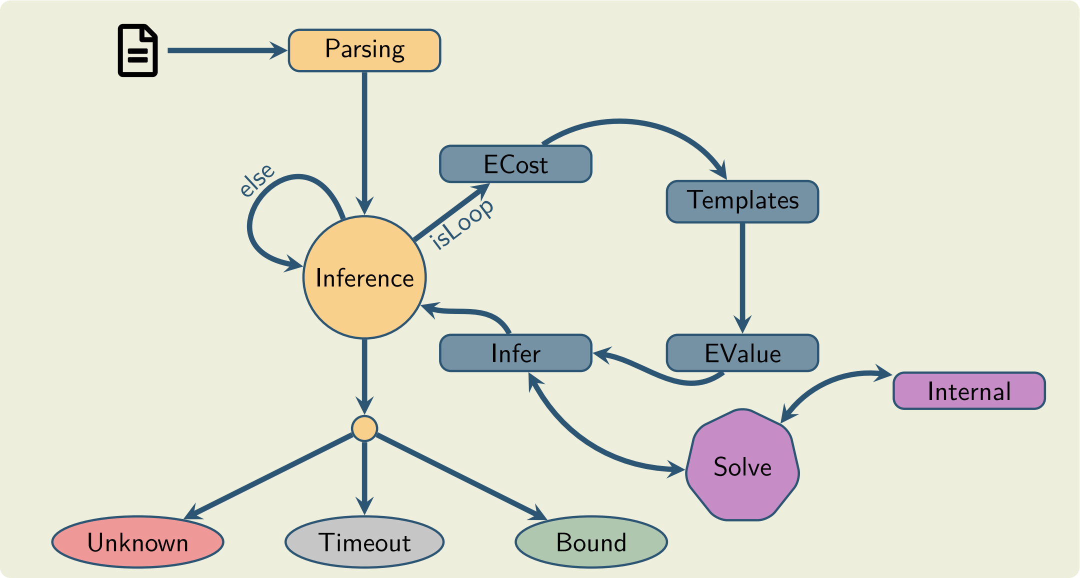
```
def geo : {
  consume(1);
  b := ber(1,1);
  if(b = 1) {
    x := x * 2;
    call geo
  } {
    skip
  }}
```

ecoimp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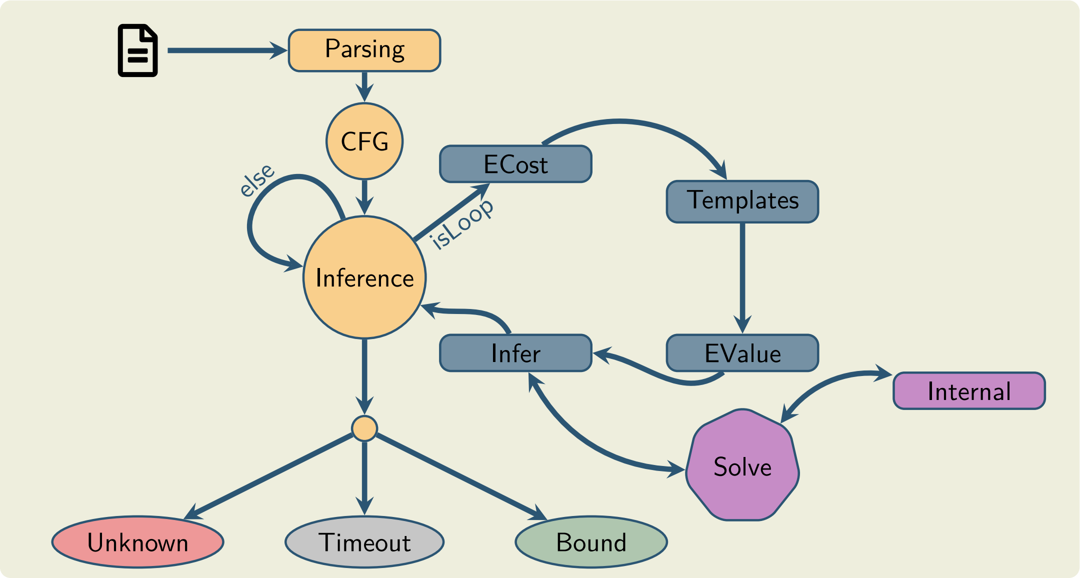
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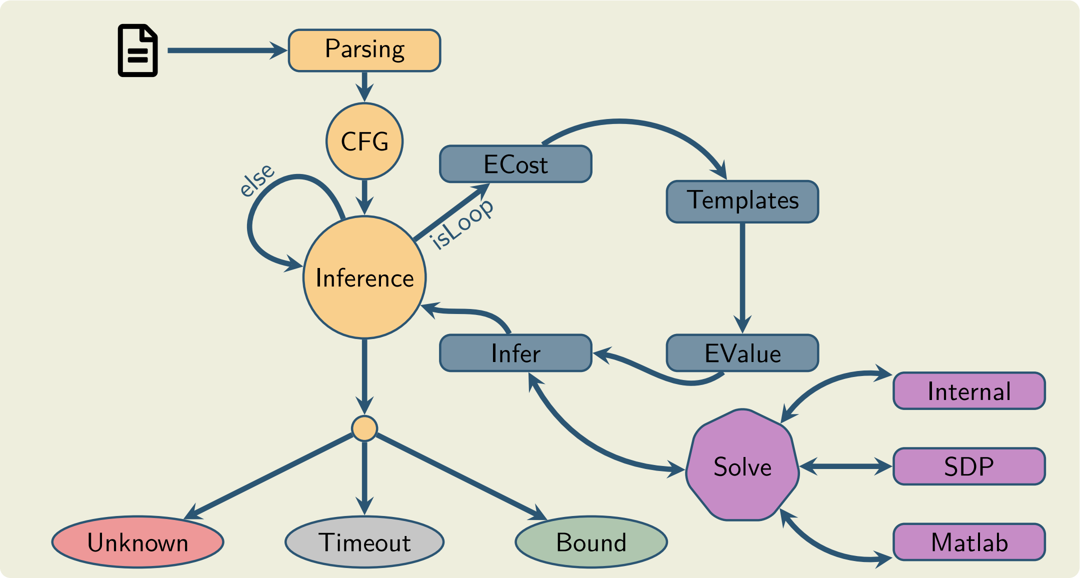

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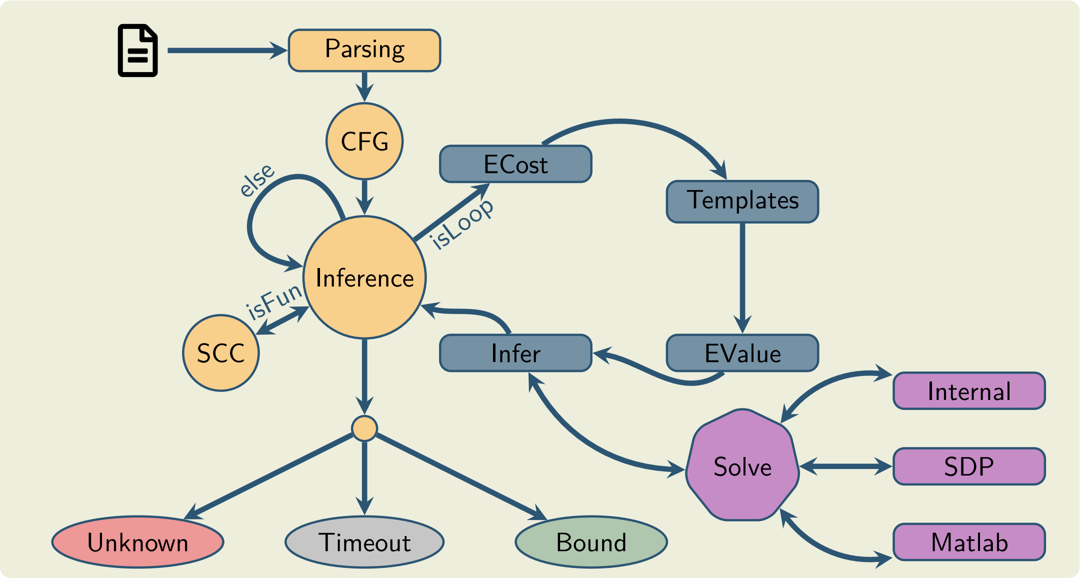
Parsing
CFG
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Templates
isFun
SCC
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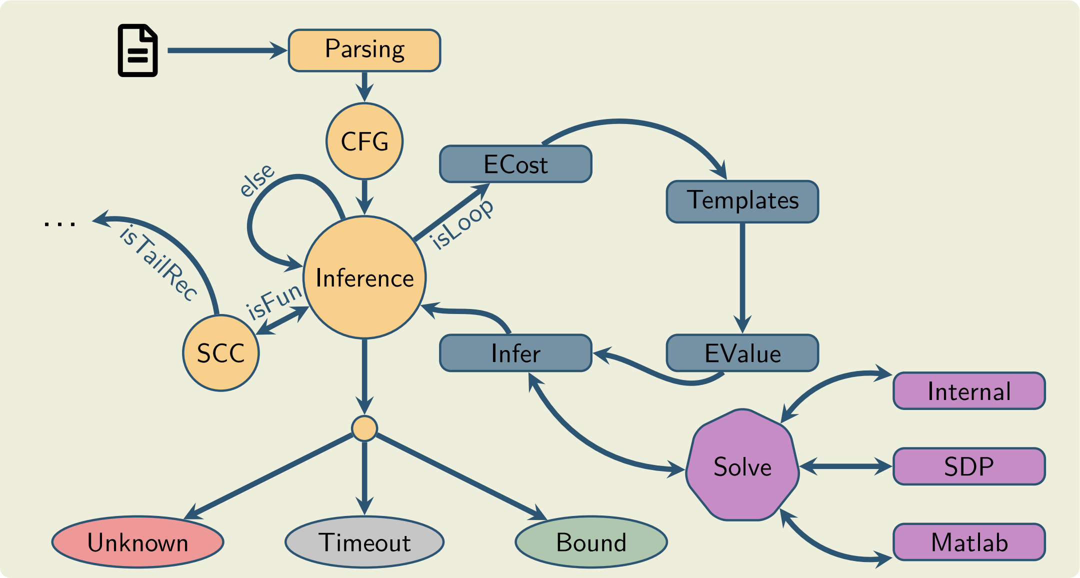
Parsing
CFG
else
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…
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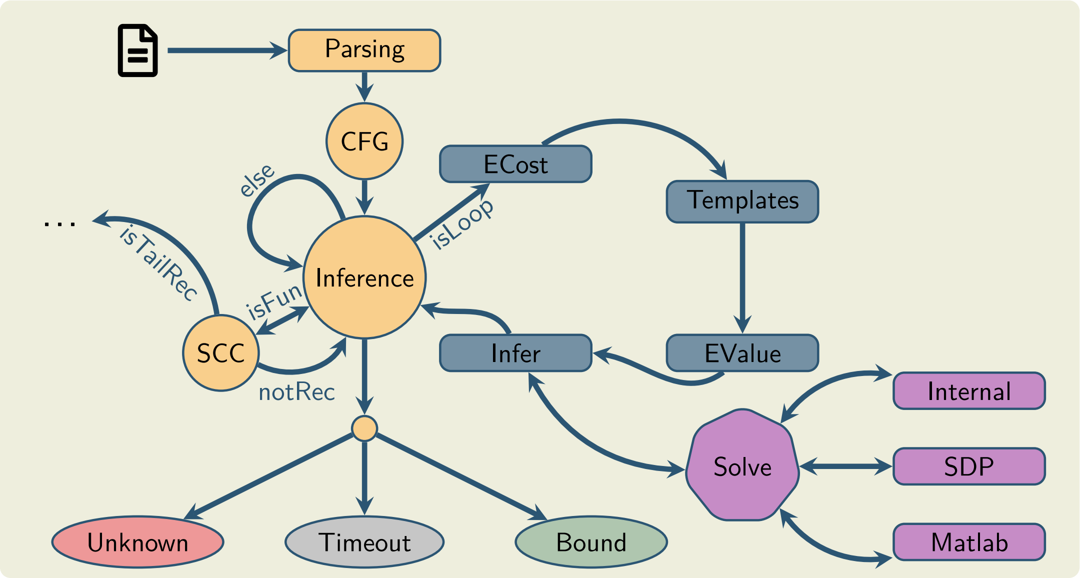

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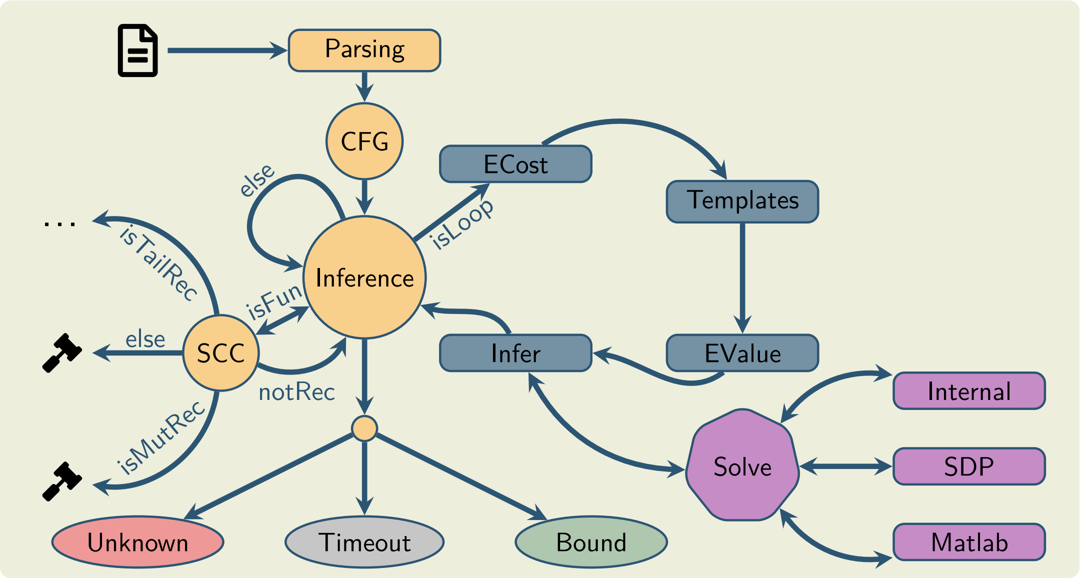
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Quickselect

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Quickselect

```
def qselect : {
  lo := 0;
  hi := N − 1;
  while(lo < hi) {
    consume(hi − lo);
    p := unif(lo, hi) ;
    if(p = pos) {
      lo := hi
    } {
      if(p < pos) {
        lo := p + 1
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Analysis

We would like to automatically derive an upper bound for the quicksort algorithm, but our initial approach can't even handle quickselect.
Our Implementation fails to solve the resulting constraints of the quickselect algorithm as equating coefficients is to weak.

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We can show positivity of a polynomial by showing that it is a sum of squares. Let p be a polynomial, then p has an SOS decomposition if

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- t…
- f…
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> NB: For a polynomial p the following holds
>
> p has an SOS decomposition $\implies p$ is positive

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- incorporate available information into constraint solving
- we maintain a context of positive polynomials in our implementation

Experiments

Problem	ecoimp		ecoimp(v1.0)		Absynth		KoAT2		Amber	
miner	.1753	✱	.0482	✔	.1274	✔	2.7567	✔		⊘
qselect	13.6977	✔	.0200	⧗		⊘		⊘		⊘
qselect_rec	14.8312	✔		⊘		⊘		⊘		⊘
coupons-10	.0754	✔	.0662	✔	32.7563	✱	.3496	⧗	.0465	✔
coupons-N	13.4197	✱	.2900	✔		⊘	.3769	⧗		⊘
pol05	25.2870	✔	.0575	✔	.3191	✔	.9076	⧗		⊘
geo	.0310	✔	.0118	✔	.0309	⧗	.5874	✔	.0461	✔
nest-4	60.1884	⧗	1.2368	✔	60.0697	⧗	1.9257	⧗		⊘
rdbub	25.4311	✔	.0569	✔	.3551	✔	.8865	⧗		⊘
complex_past	56.3168	✱	.1106	⧗	.8738	⧗	1.3813	⧗	5.1363	✔
polynomial_past_1	60.3788	⧗	.1715	⧗	.4316	⧗	1.1733	⧗	1.2191	✔

Table: Here ✔, ✱, ⧗ or ⊘ denote that a bound was found, an imprecise bound was found, no bound was found or the problem is not applicable respectively.

Summary

- static program analysis for probabilistic programs

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- abstractions via coupling

Thank you for your attention!