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Reducing Confluence of LCTRSs to Confluence of TRSs

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sum computes $\sum_{i=1}^{n} i$ for natural number n

Term Rewrite System (TRS)

$$
\begin{aligned}
\mathsf{sum}(x) &\to \mathsf{sum2}(\mathsf{geq}(0, x), x) & \mathsf{geq}(x, y) &\to \mathsf{geq2}(x, y, 0, 0) \\
\mathsf{sum2}(\mathsf{true}, x) &\to 0 & \mathsf{geq2}(\mathsf{s}(x), y, z, u) &\to \mathsf{geq2}(x, y, \mathsf{s}(z), u) \\
\mathsf{sum2}(\mathsf{false}, \mathsf{s}(x)) &\to \mathsf{plus}(\mathsf{s}(x), \mathsf{sum}(x)) & \mathsf{geq2}(\mathsf{p}(x), y, z, u) &\to \mathsf{geq2}(x, y, z, \mathsf{s}(u)) \\
\mathsf{plus}(\mathsf{s}(x), y) &\to \mathsf{s}(\mathsf{plus}(x, y)) & \mathsf{geq2}(0, \mathsf{s}(x), y, z) &\to \mathsf{geq2}(0, x, y, \mathsf{s}(z)) \\
\mathsf{plus}(\mathsf{p}(x), y) &\to \mathsf{p}(\mathsf{plus}(x, y)) & \mathsf{geq2}(0, \mathsf{p}(x), y, z) &\to \mathsf{geq2}(0, x, \mathsf{s}(y), z) \\
\mathsf{plus}(0, y) &\to y & \mathsf{geq2}(0, 0, \mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{geq2}(0, 0, x, y) \\
\mathsf{s}(\mathsf{p}(x)) &\to x & \mathsf{geq2}(0, 0, x, 0) &\to \mathsf{true} \\
\mathsf{p}(\mathsf{s}(x)) &\to x & \mathsf{geq2}(0, 0, 0, \mathsf{s}(x)) &\to \mathsf{false}
\end{aligned}
$$

Logically Constrained Term Rewrite System (LCTRS)

$$
\mathsf{sum}(x) \to 0 \ [x \leqslant 0] \qquad \mathsf{sum}(x) \to x + \mathsf{sum}(x - 1) \ [x > 0]
$$

Motivation

- Really necessary to prove known TRS criteria from scratch?
- Lift TRS confluence criteria without replaying original proof?
- Just take care of the specifics that LCTRSs have over TRSs?

Confluence Criteria

- known:
  - Kop & Nishida 2013: (weak) orthogonality
  - Winkler & Middeldorp 2018: local confluence + termination (Newman's Lemma)
  - Schöpf & Middeldorp 2023: strong and (almost) parallel closedness
- unexplored:
  (almost) development closedness, parallel critical pairs, labeling techniques, ...

How to prove/lift those in a more elegant way?

Overview

- LCTRSs
- Transformation
- Confluence

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Important Definitions

- $\mathcal{LV}\mathsf{ar}(\ell \to r\ [\varphi]) = \mathcal{V}\mathsf{ar}(\varphi) \cup (\mathcal{V}\mathsf{ar}(r) \setminus \mathcal{V}\mathsf{ar}(\ell))$
- substitution $\gamma \vDash \ell \to r\ [\varphi]$ if
  - $\mathcal{D}\mathsf{om}(\gamma) = \mathcal{V}\mathsf{ar}(\ell) \cup \mathcal{V}\mathsf{ar}(r) \cup \mathcal{V}\mathsf{ar}(\varphi)$
  - $\gamma(x) \in \mathcal{V}\mathsf{al}$ for all $x \in \mathcal{LV}\mathsf{ar}(\ell \to r\ [\varphi])$
  - $\varphi\gamma$ is valid

Example

- $\mathcal{LV}\mathsf{ar}(\mathsf{f}(x, y) \to y\ [x = u]) = \{x, u\}$
- $\mathcal{LV}\mathsf{ar}(\mathsf{f}(x, y) \to z\ [x = 1]) = \{x, z\}$

$\{x \mapsto \mathsf{g}(v, 3), y \mapsto 2, z \mapsto 3 + 2\}$ ✗

$\{x \mapsto 1, y \mapsto \mathsf{g}(v, 3), z \mapsto 5\}$ ✓

Rewrite Relation

$\mathcal{R}_{\mathsf{rc}}$ is the union of $\mathcal{R}$ and calculation rules $\mathcal{R}_{\mathsf{ca}}$

$$C[\ell\gamma] \to_{\mathsf{rc}} C[r\gamma] \qquad \text{if } \ell \to r\ [\varphi] \in \mathcal{R}_{\mathsf{rc}} \text{ and } \gamma \vDash \ell \to r\ [\varphi]$$

Example

LCTRS $\mathcal{M}$

$$\mathcal{I}_{\text{Bool}} = \mathbb{B} \qquad \mathcal{I}_{\text{Int}} = \mathbb{Z}$$

$$\mathcal{F}_{\text{te}} = \quad \ldots, \text{-1}, 0, 1, \ldots : \text{Int} \qquad \text{max} : [\text{Int}] \Rightarrow \text{Int}$$

$$\mathcal{F}_{\text{th}} = \quad \ldots, \text{-1}, 0, 1, \ldots : \text{Int} \qquad \wedge : [\text{Bool} \times \text{Bool}] \Rightarrow \text{Bool}$$

$$\text{true}, \text{false} : \text{Bool} \qquad +, - : [\text{Int} \times \text{Int}] \Rightarrow \text{Int}$$

$$\neg : [\text{Bool}] \Rightarrow \text{Bool} \qquad \leqslant, \geqslant, = : [\text{Int} \times \text{Int}] \Rightarrow \text{Bool}$$

$$\mathcal{M} = \quad \text{max}(x, y) \to x \; [x \geqslant y] \qquad \text{max}(x, y) \to y \; [y \geqslant x] \qquad \text{max}(x, y) \to \text{max}(y, x)$$

$$\text{max}(2 + 1, 1 + 3) \to \text{max}(3, 1 + 3) \to \text{max}(3, 4) \to \text{max}(4, 3) \to 4$$

Constraint Terms

pair $s\ [\varphi]$ of term s and logical constraint φ

Equivalence of Constraint Terms

$s\ [\varphi] \sim t\ [\psi]$ if $s\gamma = t\delta$ for all $\gamma \vDash \varphi$ and $\delta \vDash \psi$ (and vice versa)

Example

$\mathsf{max}(3, y)\ [y = 3] \sim \mathsf{max}(x, 3)\ [x = 3]$ $\qquad$ $\mathsf{max}(3, y)\ [y > 3] \nsim \mathsf{max}(3, y)\ [y > 4]$

$\mathsf{max}(x, y) \sim \mathsf{max}(y, x)$

Rewrite Relation on Constrained Terms

$C[\ell\gamma]\ [\varphi] \to C[r\gamma]\ [\varphi]$ $\qquad$ if $\rho\colon \ell \to r\ [\psi] \in \mathcal{R}_{\mathsf{rc}}$,

$\gamma(x) \in \mathcal{V}\mathsf{al} \cup \mathcal{V}\mathsf{ar}(\varphi)$ for all $x \in \mathcal{LV}\mathsf{ar}(\rho)$

φ is sat and $\varphi \Rightarrow \psi\gamma$ valid

rewrite relation including equivalence $\sim \cdot \to \cdot \sim$ is denoted by $\xrightarrow{\sim}$

Example

LCTRS $\mathcal{M}$

$$\mathcal{I}_{\mathsf{Bool}} = \mathbb{B} \qquad \mathcal{I}_{\mathsf{Int}} = \mathbb{Z}$$

$$\mathcal{F}_{\mathsf{te}} = \quad \ldots, \text{-}1, 0, 1, \ldots : \mathsf{Int} \qquad \mathsf{max} : [\mathsf{Int}] \Rightarrow \mathsf{Int}$$

$$\mathcal{F}_{\mathsf{th}} = \quad \ldots, \text{-}1, 0, 1, \ldots : \mathsf{Int} \qquad \wedge : [\mathsf{Bool} \times \mathsf{Bool}] \Rightarrow \mathsf{Bool}$$

$$\mathsf{true}, \mathsf{false} : \mathsf{Bool} \qquad +, - : [\mathsf{Int} \times \mathsf{Int}] \Rightarrow \mathsf{Int}$$

$$\neg : [\mathsf{Bool}] \Rightarrow \mathsf{Bool} \qquad \leqslant, \geqslant, = : [\mathsf{Int} \times \mathsf{Int}] \Rightarrow \mathsf{Bool}$$

$$\mathcal{M} = \qquad \mathsf{max}(x, y) \to x \; [x \geqslant y] \qquad \mathsf{max}(x, y) \to y \; [y \geqslant x] \qquad \mathsf{max}(x, y) \to \mathsf{max}(y, x)$$

$$\mathsf{max}(x, 1 + 3) \; [x > 4] \overset{\sim}{\to}_{\mathcal{M}} \mathsf{max}(x, 4) \; [x > 4] \to_{\mathcal{M}} x \; [x > 4]$$

Overview

- LCTRSs
- Transformation
- Confluence

Previous Work

- (weak) orthogonality, strong closedness, (almost) parallel closedness
- replaying original proofs with non-trivial adaptions
- how to continue with development closedness? (proofs terms, ...)

Key Observation

LCTRS with theory Ints

$$\mathsf{f}(x) \to y \; [x > 3] \qquad \mathsf{g}(x, y) \to \mathsf{a} \; [1 < x < 4 \wedge 3 < y < 6]$$

corresponds to infinite TRS

$$\begin{array}{ll} \mathsf{f}(4) \to 0 & \mathsf{g}(2, 4) \to \mathsf{a} \\ \mathsf{f}(4) \to 1 & \mathsf{g}(2, 5) \to \mathsf{a} \\ \ldots & \mathsf{g}(3, 4) \to \mathsf{a} \\ \mathsf{f}(5) \to 0 & \mathsf{g}(3, 5) \to \mathsf{a} \\ \ldots & \end{array}$$

Transformation

TRS $\overline{\mathcal{R}}$ transformed from LCTRS $\mathcal{R}$ consists of:

1. $\ell\tau \to r\tau$ for all $\rho\colon \ell \to r\ [\varphi] \in \mathcal{R}$ with $\tau \vDash \rho$ and $\mathcal{D}\mathsf{om}(\tau) = \mathcal{L}\mathcal{V}\mathsf{ar}(\rho)$
2. $f(v_1, \ldots, v_n) \to [\![f(v_1, \ldots, v_n)]\!]$ for all $f \in \mathcal{F}_{\mathsf{th}} \setminus \mathcal{V}\mathsf{al}$ and $v_1, \ldots, v_n \in \mathcal{V}\mathsf{al}$

Lemma

$\to_{\mathcal{R}}$ and $\to_{\overline{\mathcal{R}}}$ are the same and hence $\xrightarrow{p}_{\mathcal{R}} = \xrightarrow{p}_{\overline{\mathcal{R}}}$ for all positions p

Constrained Critical Pair (CCP)

Overlap of LCTRS $\mathcal{R}$ is $\langle \rho_1, p, \rho_2 \rangle$ with $\rho_1 \colon \ell_1 \to r_1\ [\varphi_1]$ and $\rho_2 \colon \ell_2 \to r_2\ [\varphi_2]$, satisfying:

1. ρ_1 and ρ_2 are variable-disjoint variants of rules in $\mathcal{R}_{\mathsf{rc}}$
2. $p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(\ell_2)$
3. ℓ_1 and $\ell_2|_p$ unify with mgu σ such that $\sigma(x) \in \mathcal{V}\mathsf{al} \cup \mathcal{V}$ for all $x \in \mathcal{LV}\mathsf{ar}(\rho_1) \cup \mathcal{LV}\mathsf{ar}(\rho_2)$
4. $\varphi_1\sigma \wedge \varphi_2\sigma$ is satisfiable
5. if $p = \epsilon$ then ρ_1 and ρ_2 are not variants, or $\mathcal{V}\mathsf{ar}(r_1) \nsubseteq \mathcal{V}\mathsf{ar}(\ell_1)$

$\ell_2\sigma[r_1\sigma]_p \approx r_2\sigma\ [\varphi_1\sigma \wedge \varphi_2\sigma \wedge \psi\sigma]$ is constrained critical pair with

$$\psi = \bigwedge\ \{x = x \mid x \in \mathcal{EV}\mathsf{ar}(\rho_1) \cup \mathcal{EV}\mathsf{ar}(\rho_2)\}$$

Example

$$x \approx y\ [x \geqslant y \wedge y \geqslant x] \qquad x \approx \mathsf{max}(y, x)\ [x \geqslant y] \qquad y \approx \mathsf{max}(y, x)\ [y \geqslant x]$$

Theorem

every CP $s \approx t$ of $\overline{\mathcal{R}}$ has corresponding CCP $s' \approx t'\ [\varphi']$ of $\mathcal{R}$ and substitution $\gamma \vDash \varphi'$ with $s = s'\gamma$, $t = t'\gamma$

Proof Sketch

- $s \approx t \in \overline{\mathcal{R}}$ originating from rules $\ell_1\nu \to r_1\nu$, $\ell_2\mu \to r_2\mu$
- where $\rho_1 \colon \ell_1 \to r_1\ [\varphi_1]$ and $\rho_2 \colon \ell_2 \to r_2\ [\varphi_2]$ in $\mathcal{R}_{\mathsf{rc}}$
- with $\nu \vDash \rho_1$ and $\mu \vDash \rho_2$
- show $\ell_2\delta[r_1\delta]_p \approx r_2\delta\ [\varphi\delta]$ with mgu δ forms CCP in $\mathcal{R}$

Theorem

every CP $s \approx t$ of $\overline{\mathcal{R}}$ has corresponding CCP $s' \approx t'\ [\varphi']$ of $\mathcal{R}$ and substitution $\gamma \vDash \varphi'$ with $s = s'\gamma$, $t = t'\gamma$

Converse not True in General

LCTRS with theory Ints

$$\mathsf{a} \to x\ [x = 0]$$

admits CCP

$$x \approx x'\ [x = 0 \wedge x' = 0]$$

but $\overline{\mathcal{R}}$ consisting of $\mathsf{a} \to 0$ has none
$\Longrightarrow$ orthogonality of $\overline{\mathcal{R}}$ does not imply orthogonality of $\mathcal{R}$

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Multi-Step Rewriting

multi-step relation $\mathrel{\multimap\!\!\!\rightarrow}$ on constrained terms:

1. $x\ [\varphi] \mathrel{\multimap\!\!\!\rightarrow} x\ [\varphi]$ for all variables x,
2. $f(s_1, \ldots, s_n)\ [\varphi] \mathrel{\multimap\!\!\!\rightarrow} f(t_1, \ldots, t_n)\ [\varphi]$ if $s_i\ [\varphi] \mathrel{\multimap\!\!\!\rightarrow} t_i\ [\varphi]$ for $1 \leqslant i \leqslant n$,
3. $\ell\sigma\ [\varphi] \mathrel{\multimap\!\!\!\rightarrow} r\tau\ [\varphi]$ if $\rho\colon \ell \to r\ [\psi] \in \mathcal{R}_{\mathsf{rc}}$, $\sigma(x) \in \mathcal{V}\mathsf{al} \cup \mathcal{V}\mathsf{ar}(\varphi)$ for all $x \in \mathcal{L}\mathcal{V}\mathsf{ar}(\rho)$, φ is satisfiable, $\varphi \Rightarrow \psi\sigma$ is valid, and $\sigma\ [\varphi] \mathrel{\multimap\!\!\!\rightarrow} \tau\ [\varphi]$.

with

- $\sigma\ [\varphi] \mathrel{\multimap\!\!\!\rightarrow} \tau\ [\varphi]$ denotes $\sigma(x)\ [\varphi] \mathrel{\multimap\!\!\!\rightarrow} \tau(x)\ [\varphi]$ for all variables $x \in \mathcal{D}\mathsf{om}(\sigma)$
- $\tilde{\mathrel{\multimap\!\!\!\rightarrow}} = \sim \cdot \mathrel{\multimap\!\!\!\rightarrow} \cdot \sim$

Remarks

- how to correctly merge constraints in 2?
- simplified it by unifying $\to_{\mathsf{ru}}$ and $\to_{\mathsf{ca}}$
- parallel-step relation is subsumed

Definition

$s \approx t\ [\varphi]$ is development closed if $s \approx t\ [\varphi] \overset{\sim}{\circ\!\!\!\!\rightarrow}_{\geqslant 1} u \approx v\ [\psi]$ for some trivial $u \approx v\ [\psi]$

Definition

$s \approx t\ [\varphi]$ is almost development closed if

- it is inner and development closed, or
- it is an overlay and $s \approx t\ [\varphi] \overset{\sim}{\circ\!\!\!\!\rightarrow}_{\geqslant 1} \cdot \overset{\sim}{\rightarrow}^{*}_{\geqslant 2} u \approx v\ [\psi]$ for some trivial $u \approx v\ [\psi]$

Lemma

if $s \approx t\ [\varphi] \overset{\sim}{\circ\!\!\!\!\rightarrow}_{\geqslant 1} u \approx v\ [\psi]$ then for all $\sigma \vDash \varphi$ exists $\delta \vDash \psi$ such that $s\sigma \circ\!\!\!\!\rightarrow u\delta$ and $t\sigma = v\delta$

Lemma

if $s \approx t\ [\varphi]$ is almost development closed then for all $\sigma \vDash \varphi$ we have $s\sigma \mathrel{\multimap} \cdot\ {}^*\!\!\leftarrow t\sigma$

Theorem

if LCTRS $\mathcal{R}$ is almost development closed then so is $\overline{\mathcal{R}}$

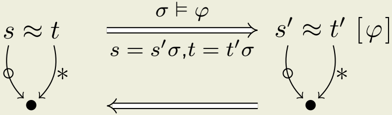
Proof Sketch

- assume $s \approx t \in \overline{\mathcal{R}}$
- there exists $s' \approx t'\ [\varphi] \in \mathcal{R}$ with $s'\sigma = s$, $t'\sigma = t$, $\sigma \vDash \varphi$
- almost development closed $\implies s = s'\sigma \mathrel{\multimap} \cdot\ {}^*\!\!\leftarrow t'\sigma = t$ or $s = s'\sigma \mathrel{\multimap} t'\sigma = t$
- $\overline{\mathcal{R}}$ is development closed

Corollary

left-linear almost development closed LCTRSs are confluent

Overview



Example

$$\mathsf{f}(x, y) \to \mathsf{g}(\mathsf{a}, y + y)\ [y \geqslant x \wedge y = 1]$$
$$\mathsf{f}(x, y) \to \mathsf{g}(\mathsf{b}, 2)\ [x \geqslant y \wedge x = 1]$$
$$\mathsf{g}(x, y) \to \mathsf{g}(y, x) \qquad \mathsf{a} \to \mathsf{b}$$

$$\mathsf{f}(1,1) \to \mathsf{g}(\mathsf{a}, 1 + 1) \qquad \mathsf{f}(0, 1) \to \mathsf{g}(\mathsf{a}, 1 + 1)$$
$$\mathsf{f}(1,1) \to \mathsf{g}(\mathsf{b}, 2) \qquad \mathsf{f}(1, 0) \to \mathsf{g}(\mathsf{b}, 2)$$
$$\mathsf{g}(x, y) \to \mathsf{g}(y, x) \qquad \mathsf{a} \to \mathsf{b} \qquad \dots$$

$$\mathsf{g}(\mathsf{a}, y + y) \approx \mathsf{g}(\mathsf{b}, 2)\ [x \geqslant y \wedge x = 1 \wedge y \geqslant x \wedge y = 1]$$

$$\mathsf{g}(\mathsf{a}, 1 + 1) \approx \mathsf{g}(\mathsf{b}, 2)$$

Theorem

if LCTRS $\mathcal{R}$ is almost development closed then so is $\overline{\mathcal{R}}$

Example

LCTRS $\mathcal{R}$ with theory Ints

$$\mathsf{f}(x) \to \mathsf{g}(x) \quad \mathsf{f}(x) \to \mathsf{h}(x)\ [1 \leqslant x \leqslant 2] \quad \mathsf{g}(x) \to \mathsf{h}(2)\ [x = 2z] \quad \mathsf{g}(x) \to \mathsf{h}(1)\ [x = 2z + 1]$$

transformed into TRS $\overline{\mathcal{R}}$

$$\begin{array}{lll} \mathsf{f}(x) \to \mathsf{g}(x) & \mathsf{f}(1) \to \mathsf{h}(1) & \mathsf{g}(n) \to \mathsf{h}(1) \quad \text{for all odd } n \in \mathbb{Z} \\ & \mathsf{f}(2) \to \mathsf{h}(2) & \mathsf{g}(n) \to \mathsf{h}(2) \quad \text{for all even } n \in \mathbb{Z} \end{array}$$

admits two (modulo symmetry) critical pairs $\mathsf{g}(1) \approx \mathsf{h}(1)$, $\mathsf{g}(2) \approx \mathsf{h}(2)$
$\Longrightarrow$ $\overline{\mathcal{R}}$ is almost development closed
$\Longrightarrow$ $\mathsf{g}(x) \approx \mathsf{h}(x)\ [1 \leqslant x \leqslant 2]$ is not almost development closed

Potential Outlook

- labeling techniques
- parallel critical pair criteria
- ...

Theorem (Toyama 1981)

left-linear TRS $\mathcal{R}$ is confluent if

- $s \mathrel{+\!\!\!\!\!\twoheadrightarrow} \cdot\ {}^*\!\!\leftarrow t$ for all $s \approx t \in \mathsf{CP}(\mathcal{R})$
- for every parallel critical peak $t\ {}^{P}\!\!\twoheadleftarrow\!\!\!\!\!+ s \to^{\epsilon} u$ exists a term v and a set P' of parallel positions such that $t \to^* v\ {}^{P'}\!\!\twoheadleftarrow\!\!\!\!\!+ u$ with $\mathcal{V}\mathsf{ar}(v, P') \subseteq \mathcal{V}\mathsf{ar}(s, P)$

Summary

- merge LCTRS rewrite relations
- multi-step (and parallel-step) rewrite relation
- much simpler approach to lift TRS confluence criteria
- (almost) development closedness