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# Confluence Criteria for Logical Constrained Rewrite Systems

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sum computes $\sum_{i=1}^{n} i$ for natural number $n$

## Term Rewrite System (TRS)

$$
\begin{aligned}
\mathsf{sum}(x) &\to \mathsf{sum2}(\mathsf{geq}(0, x), x) \\
\mathsf{sum2}(\mathsf{true}, x) &\to 0 \\
\mathsf{sum2}(\mathsf{false}, \mathsf{s}(x)) &\to \mathsf{plus}(\mathsf{s}(x), \mathsf{sum}(x)) \\
\mathsf{plus}(\mathsf{s}(x), y) &\to \mathsf{s}(\mathsf{plus}(x, y)) \\
\mathsf{plus}(\mathsf{p}(x), y) &\to \mathsf{p}(\mathsf{plus}(x, y)) \\
\mathsf{plus}(0, y) &\to y \\
\mathsf{s}(\mathsf{p}(x)) &\to x \\
\mathsf{p}(\mathsf{s}(x)) &\to x
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{geq}(x, y) &\to \mathsf{geq2}(x, y, 0, 0) \\
\mathsf{geq2}(\mathsf{s}(x), y, z, u) &\to \mathsf{geq2}(x, y, \mathsf{s}(z), u) \\
\mathsf{geq2}(\mathsf{p}(x), y, z, u) &\to \mathsf{geq2}(x, y, z, \mathsf{s}(u)) \\
\mathsf{geq2}(0, \mathsf{s}(x), y, z) &\to \mathsf{geq2}(0, x, y, \mathsf{s}(z)) \\
\mathsf{geq2}(0, \mathsf{p}(x), y, z) &\to \mathsf{geq2}(0, x, \mathsf{s}(y), z) \\
\mathsf{geq2}(0, 0, \mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{geq2}(0, 0, x, y) \\
\mathsf{geq2}(0, 0, x, 0) &\to \mathsf{true} \\
\mathsf{geq2}(0, 0, 0, \mathsf{s}(x)) &\to \mathsf{false}
\end{aligned}
$$

## Logically Constrained Term Rewrite System (LCTRS)

$$\mathsf{sum}(x) \to 0 \ [x \leqslant 0] \qquad \mathsf{sum}(x) \to x + \mathsf{sum}(x - 1) \ [x > 0]$$

# Motivation

LCTRS $\mathcal{M}$

$$\mathsf{max}(x,y) \to x\ [x \geqslant y] \qquad \mathsf{max}(x,y) \to y\ [y \geqslant x] \qquad \mathsf{max}(x,y) \to \mathsf{max}(y,x)$$

with critical pairs (ignoring symmetry)

$$x \approx y\ [x \geqslant y \land y \geqslant x] \qquad x \approx \mathsf{max}(y,x)\ [x \geqslant y] \qquad y \approx \mathsf{max}(y,x)\ [y \geqslant x]$$

confluent?

# Confluence Criteria

- known criteria:
  - Kop & Nishida 2013: (weak) orthogonality
  - Winkler & Middeldorp 2018: local confluence + termination (Newman's Lemma)
- unexplored criteria:
  strongly closedness, (almost) parallel closedness, (almost) development closedness, ...

Can those criteria be adapted towards LCTRSs?

# Overview

- Logically Constrained Term Rewrite Systems (LCTRSs)
- Confluence
- Automation

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## Important Definitions

- $\mathcal{LV}\mathsf{ar}(\ell \to r\ [\varphi]) = \mathcal{V}\mathsf{ar}(\varphi) \cup (\mathcal{V}\mathsf{ar}(r) \setminus \mathcal{V}\mathsf{ar}(\ell))$
- substitution $\gamma \vDash \ell \to r\ [\varphi]$ if
  - $\mathcal{D}\mathsf{om}(\gamma) = \mathcal{V}\mathsf{ar}(\ell) \cup \mathcal{V}\mathsf{ar}(r) \cup \mathcal{V}\mathsf{ar}(\varphi)$
  - $\gamma(x) \in \mathcal{V}\mathsf{al}$ for all $x \in \mathcal{LV}\mathsf{ar}(\ell \to r\ [\varphi])$
  - $\varphi\gamma$ is valid
- substitution $\gamma \vDash \varphi$

## Example

- $\mathcal{LV}\mathsf{ar}(\mathsf{f}(x, y) \to y\ [x = u]) = \{x, u\}$
- $\mathcal{LV}\mathsf{ar}(\mathsf{f}(x, y) \to z\ [x = \mathsf{1}]) = \{x, z\}$

$\{x \mapsto \mathsf{g}(v, \mathsf{3}), y \mapsto \mathsf{2}, z \mapsto \mathsf{3} + \mathsf{2}\}$ ✗

$\{x \mapsto \mathsf{2}, y \mapsto \mathsf{g}(v, \mathsf{3}), z \mapsto \mathsf{5}\}$ ✓

## Rewrite Relation

$\to_{\mathcal{R}}$ for LCTRS $\mathcal{R}$ is union $\to_{\mathsf{ru}}$ and $\to_{\mathsf{ca}}$:

$$C[\ell\gamma] \to_{\mathsf{ru}} C[r\gamma] \qquad \text{if } \ell \to r\ [\varphi] \in \mathcal{R} \text{ and } \gamma \vDash \ell \to r\ [\varphi]$$

$$C[f(s_1, \dots, s_n)] \to_{\mathsf{ca}} C[v] \qquad \text{if } f \in \mathcal{F}_{\mathsf{th}} \setminus \mathcal{F}_{\mathsf{te}}, s_1, \dots, s_n \in \mathcal{V}\mathsf{al}, \text{ and } v \text{ is value of } f(s_1, \dots, s_n)$$

# Example

LCTRS $\mathcal{M}$

$$\mathcal{I}_{\mathsf{Bool}} = \mathbb{B} \qquad \mathcal{I}_{\mathsf{Int}} = \mathbb{Z}$$

$$\begin{aligned}
\mathcal{F}_{\mathsf{te}} &= \quad \ldots, \text{-}1, 0, 1, \ldots : \mathsf{Int} & \mathsf{max} &: [\mathsf{Int}] \Rightarrow \mathsf{Int} \\
\mathcal{F}_{\mathsf{th}} &= \quad \ldots, \text{-}1, 0, 1, \ldots : \mathsf{Int} & \wedge &: [\mathsf{Bool} \times \mathsf{Bool}] \Rightarrow \mathsf{Bool} \\
& \quad \mathsf{true}, \mathsf{false} : \mathsf{Bool} & +, - &: [\mathsf{Int} \times \mathsf{Int}] \Rightarrow \mathsf{Int} \\
& \quad \neg : [\mathsf{Bool}] \Rightarrow \mathsf{Bool} & \leqslant, \geqslant, = &: [\mathsf{Int} \times \mathsf{Int}] \Rightarrow \mathsf{Bool}
\end{aligned}$$

$$\mathcal{M} = \quad \mathsf{max}(x, y) \to x \ [x \geqslant y] \qquad \mathsf{max}(x, y) \to y \ [y \geqslant x] \qquad \mathsf{max}(x, y) \to \mathsf{max}(y, x)$$

$$\mathsf{max}(2 + 1, 1 + 3) \to_{\mathsf{ca}} \mathsf{max}(3, 1 + 3) \to_{\mathsf{ca}} \mathsf{max}(3, 4) \to_{\mathsf{ru}} \mathsf{max}(4, 3) \to_{\mathsf{ru}} 4$$

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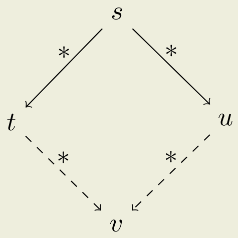
## Confluence




## Critical Pairs (CPs)

- (weak) orthogonality
- Knuth-Bendix criterion
- strongly closed
- (almost) parallel closed
- (almost) development closed
- labeling techniques
- CP closing systems
- commutation based methods
- ...

# Constrained Critical Pair (CCP)

Overlap of LCTRS $\mathcal{R}$ is $\langle \rho_1, p, \rho_2 \rangle$ with $\rho_1 \colon \ell_1 \to r_1\ [\varphi_1]$ and $\rho_2 \colon \ell_2 \to r_2\ [\varphi_2]$, satisfying:

1. $\rho_1$ and $\rho_2$ are variable-disjoint variants of rules in $\mathcal{R} \cup \mathcal{R}_{\mathsf{ca}}$
2. $p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(\ell_2)$
3. $\ell_1$ and $\ell_2|_p$ unify with mgu $\sigma$ such that $\sigma(x) \in \mathcal{V}\mathsf{al} \cup \mathcal{V}$ for all $x \in \mathcal{LV}\mathsf{ar}(\rho_1) \cup \mathcal{LV}\mathsf{ar}(\rho_2)$
4. $\varphi_1\sigma \wedge \varphi_2\sigma$ is satisfiable
5. if $p = \epsilon$ then $\rho_1$ and $\rho_2$ are not variants, or $\mathcal{V}\mathsf{ar}(r_1) \nsubseteq \mathcal{V}\mathsf{ar}(\ell_1)$

$\ell_2\sigma[r_1\sigma]_p \approx r_2\sigma\ [\varphi_1\sigma \wedge \varphi_2\sigma \wedge \psi\sigma]$ is constrained critical pair with

$$\psi = \bigwedge\ \{x = x \mid x \in \mathcal{EV}\mathsf{ar}(\rho_1) \cup \mathcal{EV}\mathsf{ar}(\rho_2)\}$$

## Example

$$x \approx y\ [x \geqslant y \wedge y \geqslant x] \qquad x \approx \mathsf{max}(y, x)\ [x \geqslant y] \qquad y \approx \mathsf{max}(y, x)\ [y \geqslant x]$$

# Constrained Terms

- $s\ [\varphi]$ with term $s$ and logical constraint $\varphi$
- $s\ [\varphi] \sim t\ [\psi]$ if for all $\gamma \vDash \varphi$ exists $\tau \vDash \psi$ with $s\gamma = t\tau$ (and vice versa)
- important for (confluence) analysis
- $\overset{\sim}{\to}{}^{*}_{\mathcal{R}} := \sim \cdot \to_{\mathsf{ru}} \cup \to_{\mathsf{ca}} \cdot \sim$

## Example

LCTRS $\mathcal{M}$

$$\mathsf{max}(x, y) \to x\ [x \geqslant y] \qquad \mathsf{max}(x, y) \to y\ [y \geqslant x] \qquad \mathsf{max}(x, y) \to \mathsf{max}(y, x)$$

rewrite constrained terms

$$\mathsf{max}(x, x)\ [x \geqslant 4] \to_{\mathsf{ru}} x\ [x \geqslant 4] \qquad // \ x \geqslant 4 \Rightarrow x \geqslant x$$

$$\mathsf{max}(x + y, 1)\ [x = 2 \wedge y \geqslant 0] \to_{\mathsf{ca}} \mathsf{max}(z, 1)\ [x = 2 \wedge y \geqslant 0 \wedge z = x + y]$$

$$\mathsf{max}(x, 1)\ [x = 2] \sim \mathsf{max}(2, 1) \to_{\mathsf{ru}} 2$$

# Trivial CCP

CCP $s \approx t\ [\varphi]$ is trivial if $s\sigma = t\sigma$ for every $\sigma \vDash \varphi$

## Joinability of CCPs

For $s \approx t\ [\varphi]$ cannot simply be defined as $s\ [\varphi] \xrightarrow{\sim}{}^*_{\mathcal{R}} \cdot {}^*_{\mathcal{R}}\xleftarrow{\sim} t\ [\varphi]$. Terminating LCTRS $\mathcal{R}$:

$$\mathsf{f}(x, y) \to \mathsf{g}(x, 1+1) \qquad\qquad \mathsf{h}(\mathsf{f}(x, y)) \to \mathsf{h}(\mathsf{g}(y, 1+1))$$

with CCP $\mathsf{h}(\mathsf{g}(x, 1+1)) \approx \mathsf{h}(\mathsf{g}(y, 1+1))$ for which

$$\mathsf{h}(\mathsf{g}(x, 1+1)) \to_{\mathsf{ca}} \mathsf{h}(\mathsf{g}(x, z))\ [z = 1+1] \sim \mathsf{h}(\mathsf{g}(y, v))\ [v = 1+1]\ {}_{\mathsf{ca}}\!\leftarrow \mathsf{h}(\mathsf{g}(y, 1+1))$$

but

$$\mathsf{h}(\mathsf{g}(1,2))\ {}^*\!\leftarrow \mathsf{h}(\mathsf{f}(1,2)) \to^* \mathsf{h}(\mathsf{g}(2,2))$$

## Joinability

- constrained equation $s \approx t\ [\varphi]$ is trivial if $s\sigma = t\sigma$ for any $\sigma \vDash \varphi$
- critical pair $s \approx t\ [\varphi]$ is joinable if $s \approx t\ [\varphi] \xrightarrow{\sim}{}^{*}_{\mathcal{R}} u \approx v\ [\psi]$ for trivial $u \approx v\ [\psi]$

### Example cont'd

$$
\begin{aligned}
&\mathsf{h}(\mathsf{g}(x, 1+1)) \approx \mathsf{h}(\mathsf{g}(y, 1+1)) \\
&\quad \to_{\mathsf{ca}} \mathsf{h}(\mathsf{g}(x, v)) \approx \mathsf{h}(\mathsf{g}(y, 1+1))\ [v = 1+1] \\
&\quad \to_{\mathsf{ca}} \mathsf{h}(\mathsf{g}(x, v)) \approx \mathsf{h}(\mathsf{g}(y, z))\ [v = 1+1 \wedge z = 1+1]
\end{aligned}
$$

BUT $\sigma = \{v \mapsto 2, z \mapsto 2\}$ and $\sigma \vDash v = 1+1 \wedge z = 1+1$ thus $\mathsf{h}(\mathsf{g}(x, 2)) \neq \mathsf{h}(\mathsf{g}(y, 2))$.

# Adding Extra Variables

$inp$ models user input: $\mathsf{start}(x_1, \ldots, x_n) \to \mathsf{comp}(x_1, \ldots, x_n, inp)$

## Example

LCTRS $\mathcal{R}$

$$\mathsf{f}(x) \to \mathsf{g}(y) \qquad\qquad \mathsf{g}(y) \to \mathsf{a}\ [y = y]$$

with joinable CCP $\mathsf{g}(y) \approx \mathsf{g}(y')\ [y = y \wedge y' = y']$. $\mathcal{R}$ is terminating and hence confluent.
Without trivial equations $\psi$ we have CCP $\mathsf{g}(y) \approx \mathsf{g}(y')\ [\mathsf{true}]$ giving wrongly non-confluence.

## Strongly Closed (TRS)

CP $s \approx t$ is strongly closed if $s \to^{=} \cdot {}^{*}\!\leftarrow t$ and $s \to^{*} \cdot {}^{=}\!\leftarrow t$.

## Strongly Closed

CCP $s \approx t\ [\varphi]$ is strongly closed if

1. $s \approx t\ [\varphi] \stackrel{\sim}{\to}^{*}_{\geqslant 1} \cdot \stackrel{\sim}{\to}^{=}_{\geqslant 2} u \approx v\ [\psi]$ for some trivial $u \approx v\ [\psi]$, and
2. $s \approx t\ [\varphi] \stackrel{\sim}{\to}^{*}_{\geqslant 2} \cdot \stackrel{\sim}{\to}^{=}_{\geqslant 1} u \approx v\ [\psi]$ for some trivial $u \approx v\ [\psi]$.

## Theorem (Huet 1980)

linear TRS is strongly confluent if all its critical pairs are strongly closed

## Theorem

linear LCTRS is strongly confluent if all its constrained critical pairs are strongly closed

## Example

LCTRS $\mathcal{M}$ with CCPs:

$$x \approx y \ [x \geqslant y \wedge y \geqslant x] \qquad x \approx \mathsf{max}(y,x) \ [x \geqslant y] \qquad y \approx \mathsf{max}(y,x) \ [y \geqslant x]$$

not confluent by any previously existing criterion but

$$x \approx \mathsf{max}(y,x) \ [x \geqslant y] \ \xrightarrow{\geqslant 2} \ x \approx x \ [x \geqslant y]$$
$$y \approx \mathsf{max}(y,x) \ [y \geqslant x] \ \xrightarrow{\geqslant 2} \ y \approx y \ [y \geqslant x]$$

linear & strongly closed, thus confluent

## Parallel Rewriting

For LCTRS $\mathcal{R}$ the relation $\mathrel{+\!\!\!\!\!\rightarrow}_{\mathcal{R}}$ is defined as

1. $x \mathrel{+\!\!\!\!\!\rightarrow}_{\mathcal{R}} x$ for all variables $x$,
2. $f(s_1, \ldots, s_n) \mathrel{+\!\!\!\!\!\rightarrow}_{\mathcal{R}} f(t_1, \ldots, t_n)$ if $s_i \mathrel{+\!\!\!\!\!\rightarrow}_{\mathcal{R}} t_i$ for all $1 \leqslant i \leqslant n$,
3. $\ell\sigma \mathrel{+\!\!\!\!\!\rightarrow}_{\mathcal{R}} r\sigma$ with $\ell \to r\ [\varphi] \in \mathcal{R}$ and $\sigma \vDash \ell \to r\ [\varphi]$,
4. $f(v_1, \ldots, v_n) \mathrel{+\!\!\!\!\!\rightarrow} v$ with $f \in \mathcal{F}_{\mathsf{th}} \setminus \mathcal{V}\mathsf{al}$, $v_1, \ldots, v_n \in \mathcal{V}\mathsf{al}$ and $v = [\![f(v_1, \ldots, v_n)]\!]$.

## Parallel Closed

critical pair $s \approx t\ [\varphi]$ is parallel closed if

$$s \approx t\ [\varphi] \mathrel{\tilde{+\!\!\!\!\!\rightarrow}}_{\geqslant 1} u \approx v\ [\psi]$$

for some trivial $u \approx v\ [\psi]$

## Theorem

left-linear parallel-closed LCTRSs are confluent

## Example

LCTRS $\mathcal{R}$

$$\mathsf{f}(x,y) \to \mathsf{g}(\mathsf{a}, y+y)\ [y \geqslant x \wedge y = 1] \qquad \mathsf{a} \to \mathsf{b}$$

$$\mathsf{h}(\mathsf{f}(x,y)) \to \mathsf{h}(\mathsf{g}(\mathsf{b},2))\ [x \geqslant y] \qquad \mathsf{g}(x,y) \to \mathsf{g}(y,x)$$

single CP

$$\mathsf{h}(\mathsf{g}(\mathsf{a}, y+y)) \approx \mathsf{h}(\mathsf{g}(\mathsf{b},2))\ [x \geqslant y \wedge y \geqslant x \wedge y = 1]$$

$$\widetilde{\nrightarrow}^{*}_{\geqslant 1}\ \mathsf{h}(\mathsf{g}(\mathsf{b},2)) \approx \mathsf{h}(\mathsf{g}(\mathsf{b},2))\ [\mathsf{true}]$$

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## Syntactic Equality is too Weak

$$\mathsf{g}(z)\ [z = 3] \neq \mathsf{g}(3) \qquad\qquad \mathsf{g}(z)\ [z = 3] \sim \mathsf{g}(3)$$

## Triviality Check

formula $T(s,t,\varphi)$ for constrained equation $s \approx t\ [\varphi]$

$$T(s,t,\varphi) = \begin{cases} \text{true} & \text{if } s = t \\ s = t & \text{if } s,t \in \mathcal{V}\text{al} \cup \mathcal{V}\text{ar}(\varphi) \\ \displaystyle\bigwedge_{i=1}^{n} T(s_i,t_i,\varphi) & \text{if } s = f(s_1,\ldots,s_n) \text{ and } t = f(t_1,\ldots,t_n) \\ \text{false} & \text{otherwise} \end{cases}$$

## Lemma

constrained equation $s \approx t\ [\varphi]$ is trivial if and only if the formula $\varphi \implies T(s,t,\varphi)$ is valid

How to compute equivalences for $\mathcal{R}$? $\xrightarrow{\sim}{}^*_{\mathcal{R}} \cdot {}^*_{\mathcal{R}}\xleftarrow{\sim}$ is implemented as $\rightarrow^*_{\mathsf{tf}(\mathcal{R})} \cdot \sim \cdot {}_{\mathsf{tf}(\mathcal{R})}^*\leftarrow$.

## Transformation

Let $\mathcal{R}$ be an LCTRS and $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$:

$$\mathsf{tf}(t) = \begin{cases} (t, \mathsf{true}) & \text{if } t \in \mathcal{V} \\ (z, z = t) & \text{if } t \in \mathcal{V}\mathsf{al} \text{ and } z \text{ is a fresh variable} \\ (f(s_1, \dots, s_n), \varphi_1 \wedge \cdots \wedge \varphi_n) & \text{if } t = f(t_1, \dots, t_n) \text{ and } \mathsf{tf}(t_i) = (s_i, \varphi_i) \end{cases}$$

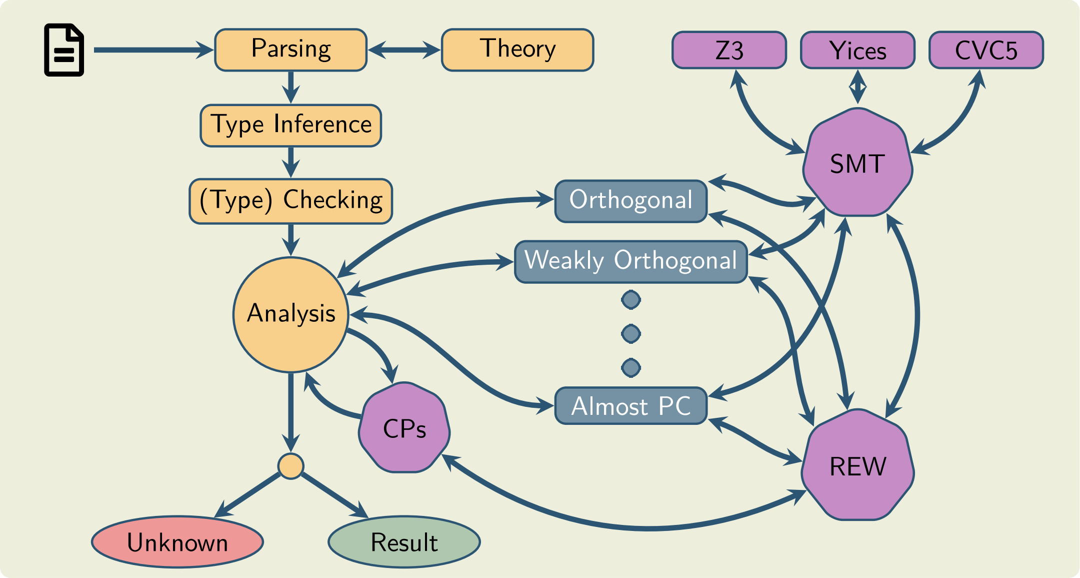
$$\mathsf{tf}(\mathcal{R}) = \{\ell' \rightarrow r\ [\varphi \wedge \psi] \mid \ell \rightarrow r\ [\varphi] \in \mathcal{R} \text{ and } \mathsf{tf}(\ell) = (\ell', \psi)\}$$

## Example

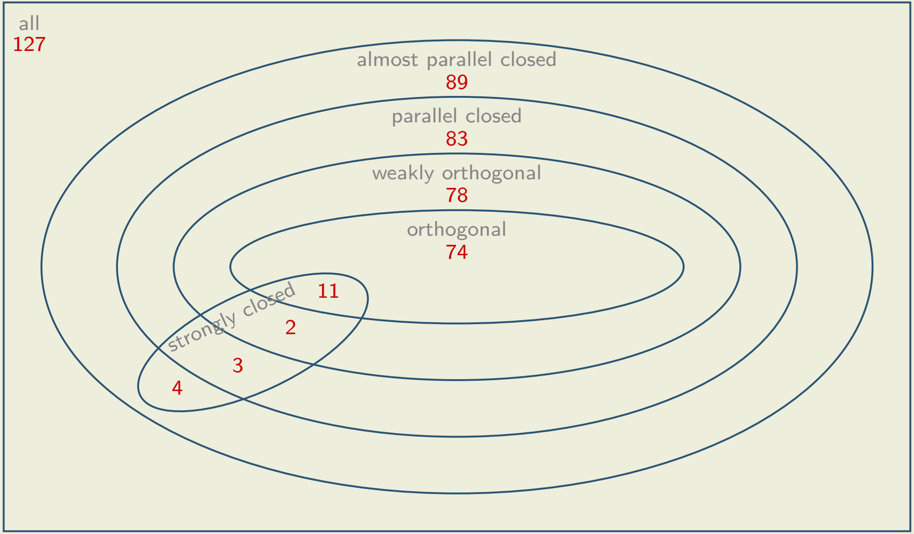
$$\mathsf{f}(x) \rightarrow z\ [z = 3] \qquad \mathsf{g}(\mathsf{f}(x)) \rightarrow \mathsf{a} \qquad \mathsf{g}(3) \rightarrow \mathsf{a} \qquad (\mathcal{R})$$

CCP $\mathsf{g}(z) \approx \mathsf{a}\ [z = 3]$ is joinable with $\mathsf{tf}(\mathcal{R})$ but not $\mathcal{R}$ without initial equivalence

# Rough Workflow of Our Tool



# Experiments



## Summary

- logical constrained rewrite systems
- strong closedness & (almost) parallel closedness for LCTRSs
- automation in tool crest

## Outlook

- more elegant & simpler approach (IWC 2023)
- How to resolve encountered problems?
- competition category in CoCo
- raise interest for LCTRSs

## Critical Pairs (CPs)

- (weak) orthogonality
- Knuth-Bendix criterion
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- (almost) parallel closed
- (almost) development closed
- labeling techniques
- CP closing systems
- commutation based methods
- ...

## Parallel CPs (PCPs)

- criterion Gramlich 96
- criterion Toyama 81
- criteria Shintani et al. 22
- labeling techniques
- commutation based
- CP closing systems

## Simultaneous CPs (SCPs)

- criterion Okui 98

## Transformation

redundant rules, order-sorted decomposition, reduction method, ...